SPECTRAL DENSITIES OF DISCRETE AND CONTINUOUS-INDEXED RANDOM FIELDS

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Abstract

This text first looks at sequences of discrete-indexed random fields. When these random fields satisfy certain linear dependence conditions uniformly, each will have a spectral density function (not necessarily continuous) that is bounded between two positive constants. These spectral density functions will converge in a weak sense to another function (not necessarily continuous) that is also bounded between two positive constants. Two examples will also be given that show the weak form of convergence seems to be the best one can get. An extra condition on the sequence will also be given which will ensure each spectral density function is continuous and that they uniformly converge to a continuous function.

Continuous-indexed random fields will then be investigated, and linear dependence coefficients specifically for such random fields will be defined. When a selection of these linear dependence conditions are satisfied, the random field will have a continuous spectral density function. Showing this involves the construction of a special class of random fields using a standard Poisson process and the original random field.

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CHAPTER 1

Introduction and Definitions

1. Introduction

When one is considering a stochastic process or time series, the covariance function (sometimes referred to as the autocovariance function) is used to study the pattern of the process as it moves through the index set, which usually is time. The spectral density function (when it exists) is the Fourier transform of the covariance function and helps when one is studying frequency properties of the process [**8**]. The estimation of the covariance and spectral density functions under various conditions of weak dependence has been studied extensively. The continuity and positivity of the spectral density function is closely connected with certain linear dependence coefficients, and plays a significant role in the spectral density estimation. This work will first look at questions about the existence of positive and continuous spectral densities for general multi-dimensional discrete-indexed (on $\mathbb{Z}^d$ for some positive integer d) stochastic processes (called random fields) satisfying certain weak dependence conditions. Then, it will delve into continuous spectral densities for continuous-indexed (on $\mathbb{R}^d$ for some positive integer d) random fields under certain other weak dependence conditions.

This chapter includes basic definitions while the next chapter will state many lemmas and theorems that are essential background material. The third chapter will be devoted to results for weakly dependent random fields of discrete index. In Chapter 3, there are two results of Bradley ([**4**], [**5**]) in which necessary and sufficient conditions on random fields are given for the existence of a continuous spectral density, and for a spectral density (not necessarily continuous) bounded between two positive constants. Two other results follow, which involve sequences of random fields satisfying the same conditions uniformly. The first of the two involving sequences was done by Bradley [**5**], and the second result involving sequences was done by the author [**19**].

Chapter 4 introduces random fields indexed by a continuum and gives the main result (Theorem 4.1). The proof of Theorem 4.1 will span Chapters 4 through 7. Most of the definitions and lemmas for discrete-indexed random fields extend nicely by trading sums for integrals and cardinality for

Lebesgue measure. However, the existence of a continuous spectral density for a random field of continuous-index is not an easy extension. The last four chapters are used for that purpose. Many of the methods and ideas used in those chapters come from Curtis Miller [**14**].

2. Definitions

The setting of this text will be on a probability space $(\Omega, \mathcal{F}, P)$, in which Ω is the sample space, $\mathcal{F}$ is a σ-field on Ω (which includes all the "events"), and P is a probability measure on $(\Omega, \mathcal{F})$. A random variable X is a real or complex valued $\mathcal{F}$-measurable function defined on Ω. A "list" of random variables $\{X_t\}$ is called a stochastic process where the index t is discrete or continuum valued. This is usually denoted by $(X_t : t \in V)$ where V is either $\mathbb{Z}$ or $\mathbb{R}$. A stochastic process whose index set is multi-dimensional (V^d for some integer $d \geq 2$, where $V = \mathbb{Z}$ or $\mathbb{R}$) is called a *random field*. For a random field $(X_\nu : \nu \in \mathbb{R}^d)$ on a probability space $(\Omega, \mathcal{F}, P)$, it is understood that the function $(\nu, \omega) \mapsto X_\nu(\omega)$ for $(\nu, \omega) \in \mathbb{R}^d \times \Omega$ is measurable with respect to the product σ-field $\mathcal{R}^d \times \mathcal{F}$ where $\mathcal{R}^d$ is the Borel σ-field on $\mathbb{R}^d$.

Weakly Stationary random fields appear many times throughout the text, and are defined below. For $k := (k_1, k_2, \ldots, k_d) \in \mathbb{R}^d$ and $\ell := (\ell_1, \ell_2, \ldots, \ell_d) \in \mathbb{R}^d$, let $k - \ell := (k_1 - \ell_1, k_2 - \ell_2, \ldots, k_d - \ell_d)$.

Definition 1.1. *Let d be a positive integer, and V be either $\mathbb{Z}$ or $\mathbb{R}$. A complex valued random field $X := (X_k : k \in V^d)$ is weakly stationary if it has the following three properties:*

(1) $E|X_k|^2 < \infty$ for all $k \in V^d$

(2) There exists an $m \in \mathbb{C}$ such that $EX_k = m$ for all $k \in V^d$

(3) There exists a function $\gamma : V^d \to \mathbb{C}$ such that for every $k, \ell \in V^d$, $\mathrm{E}(X_k - m)(\overline{X_\ell - m}) = \gamma(k - \ell)$

In the setting of Definition 1.1, a complex valued random field is one such that for each $k \in V^d$, $X_k(\omega) \in \mathbb{C}$ for each $\omega \in \Omega$. Also, if $m = 0$, then the random field is said to be *CCWS* (centered, complex, and weakly stationary). The function γ will be referred to as the *covariance function*.

Let T denote the unit circle in the complex plane, and $m(\cdot)$ denote normalized one-dimensional Lebesgue measure on T so that $m(T) = 1$. The d-dimensional product measure on T^d will be denoted $m_d(\cdot)$. Each $t = (t_1, t_2, \ldots, t_d) \in T^d$ is assigned a specific $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in (-\pi, \pi]^d$ where $t_j = e^{i\lambda_j}$ for $j = 1, 2, \ldots, d$. For every $k \in \mathbb{Z}^d$ and $t \in T^d$, define the notation $t^k := \prod_{j=1}^d t_j^{k_j} = e^{i\lambda \cdot k}$ where $\lambda \cdot k$ is the dot product. Notice that with this notation, $t^k \in T$ whenever $t \in T^d$ and $k \in \mathbb{Z}^d$.

Definition 1.2. *A Borel measurable, non-negative integrable function f on T^d is a spectral density for a CCWS random field $X := (X_k : k \in \mathbb{Z}^d)$ if for all $k \in \mathbb{Z}^d$,*

$$\gamma(k) = EX_k\overline{X}_0 = \int_{T^d} t^k f(t) dm_d(t).$$

Remark 1.1. *It will be convenient to write $\gamma(0)$ or X_0 instead of $\gamma((0,0,\dots,0))$ or $X_{(0,0,\dots,0)}$, where the 0 will be understood as the origin in $\mathbb{R}^d$. It will also be convenient to let $X_1 := X_{(1,1,\dots,1)}$ for $(1,1,\dots,1) \in \mathbb{R}^d$.*

In the continuous-index case, the spectral density function is defined over all $\mathbb{R}^d$. In this context, $dm_d(x)$ is understood to be $(2\pi)^{-d}dx$ (where dx denotes Lebesgue measure on $\mathbb{R}^d$), which is in a spirit similar to that of the discrete-index case.

Definition 1.3. *A Borel measurable, non-negative integrable function g on $\mathbb{R}^d$ is a spectral density for a CCWS random field $X := (X_\nu : \nu \in \mathbb{R}^d)$ if for all $\nu \in \mathbb{R}^d$,*

$$\gamma(\nu) = EX_\nu\overline{X}_0 = \int_{\mathbb{R}^d} e^{i\,\lambda\cdot\nu} g(\lambda) dm_d(\lambda).$$

Since an integrable function on T^d or $\mathbb{R}^d$ is uniquely determined almost everywhere by its Fourier coefficients, the spectral density function is unique if one disregards sets of Lebesgue measure zero. One can use either the spectral density function or the covariance function to describe their underlying weakly stationary process. Both contain the same information, but are complimentary to one another by expressing this information in different ways [**8**].

The following notation will be used throughout the rest of this text. For non-empty sets $Q, S \subset V^d$ where $V = \mathbb{Z}$ or $\mathbb{R}$, $\text{dist}(Q,S) := \min_{q\in Q, s\in S} \|q - s\|$, where $\|\cdot\|$ is the Euclidean distance. The cardinality of a set S is denoted $|S|$. For a fixed positive integer d, $\lambda(\cdot)$ denotes Lebesgue measure on $\mathbb{R}^d$. The L^p norm will be denoted by $\|\cdot\|_p$, where p will be either 1 or 2 in this text. Although it shouldn't cause confusion, it is good to note the difference between the λ denoting Lebesgue measure and the λ used prior to Definition 1.2.

The next few definitions are for measures of linear dependence. The subscript D (for "discrete") will be used to signify measures of linear dependence specifically for random fields of discrete index, although these measures can be used for both discrete and continuous-indexed random fields.

DEFINITION 1.4. *Let V be either $\mathbb{Z}$ or $\mathbb{R}$ and $X := (X_k : k \in V^d)$ be a CCWS random field. For any non-empty, finite, disjoint sets Q, $S \subset V^d$, define the number*

$$\mathcal{R}_D(Q,S) = \sup \frac{\left|EU\overline{W}\right|}{\|U\|_2\|W\|_2}, \tag{1.1}$$

where the supremum is taken over all pairs of complex-valued random variables U and W of the form

$$U = \sum_{k\in Q} a_k X_k \ \ \text{and} \ \ W = \sum_{k \in S} a_k X_k,$$

where $a_k \in \mathbb{C}$ for all $k \in Q \cup S$. In (1.1), and all the equations below, $0/0$ will be interpreted as 0. The linear dependence coefficients are defined for random fields in three different parts below.

For each $n \in \mathbb{N}$, define

$$\begin{aligned} q_D(X,n) = q_D(n) &:= \sup \frac{\left|E\left(\sum_{k\in Q} X_k\right)\left(\overline{\sum_{k\in S} X_k}\right)\right|}{\left\|\sum_{k\in Q} X_k\right\|_2 \left\|\sum_{k\in S} X_k\right\|_2}, && (1.2)\\ r_D(X,n) = r_D(n) &:= \sup \mathcal{R}_D(Q,S), && (1.3)\end{aligned}$$

where each supremum is taken over all non-empty, finite sets Q and $S \subset V^d$ with the property that

$$\left.\begin{array}{c} \text{there exists } u \in \{1,2,\ldots,d\} \text{ such that} \\ Q \subset \{(k_1,k_2,\ldots,k_d) \in V^d : k_u \le 0\} \\ S \subset \{(k_1,k_2,\ldots,k_d) \in V^d : k_u \ge n\}. \end{array}\right\} \tag{1.4}$$

Again, for each $n \in \mathbb{N}$, define

$$\begin{aligned} q'_D(X,n) = q'_D(n) &:= \sup \frac{\left|E\left(\sum_{k\in Q} X_k\right)\left(\overline{\sum_{k\in S} X_k}\right)\right|}{\left\|\sum_{k\in Q} X_k\right\|_2 \left\|\sum_{k\in S} X_k\right\|_2}, && (1.5)\\ r'_D(X,n) = r'_D(n) &:= \sup \mathcal{R}_D(Q,S), && (1.6)\\ \zeta_D(X,n) = \zeta_D(n) &:= \sup \frac{\left|E\left(\sum_{k\in Q} X_k\right)\left(\overline{\sum_{k\in S} X_k}\right)\right|}{|Q \cup S|} && (1.7)\end{aligned}$$

where each supremum is taken over all non-empty, finite sets Q and $S \subset V^d$ with the property that

$$\left.\begin{array}{c} \text{there exists } u \in \{1,2,\ldots,d\} \text{ and non-empty sets} \\ Q_0, S_0 \subset V \text{ with } \operatorname{dist}(Q_0,S_0) \ge n \text{ such that} \\ Q \subset \{(k_1,k_2,\ldots,k_d) \in V^d : k_u \in Q_0\} \\ S \subset \{(k_1,k_2,\ldots,k_d) \in V^d : k_u \in S_0\}. \end{array}\right\} \tag{1.8}$$

Finally, for each $n \in \mathbb{N}$, define

$$q_D^*(X,n) = q_D^*(n) := \sup \frac{\left| E\left(\sum_{k\in Q} X_k\right)\left(\overline{\sum_{k\in S} X_k}\right)\right|}{\left\|\sum_{k\in Q} X_k\right\|_2 \left\|\sum_{k\in S} X_k\right\|_2}, \tag{1.9}$$

$$r_D^*(X,n) = r_D^*(n) := \sup \mathcal{R}_D(Q,S), \tag{1.10}$$

where each supremum is taken over all non-empty, finite sets Q and $S \subset V^d$ such that $\operatorname{dist}(Q,S) \geq n$.

Notice the main difference between (1.4) and (1.8) is the interlacing of sets allowed in (1.8). When CCWS continuous-indexed random fields were studied by Miller in connection with a spectral density function [**14**], definitions similar but stronger than (1.3) and (1.10) were used. This was not very satisfying, so the definitions above were modified in a natural way to better suit random fields of continuous-index. These definitions are analogous to the above definitions (now with c as a subscript, and integrals replacing sums).

Definition 1.5. *Let $X := (X_\nu : \nu \in \mathbb{R}^d)$ be a CCWS random field. For any non-empty, disjoint, bounded Borel sets Q, $S \subset \mathbb{R}^d$, define the number*

$$\mathcal{R}_c(Q,S) = \sup \frac{\left|EU\overline{W}\right|}{\|U\|_2 \|W\|_2}, \tag{1.11}$$

where the supremum is taken over all pairs of complex-valued random variables U and W of the form

$$U = \int_Q j(\nu) X_\nu d\nu \ \text{ and } \ W = \int_S j(\nu) X_\nu d\nu,$$

where $j(\nu)$ is a bounded, complex valued Borel function. Again, in (1.11) and the equations below, $0/0$ *will be interpreted as 0.*

For each $s \in \mathbb{R}_+$, define

$$q_c(X,s) = q_c(s) := \sup \frac{\left| E\left(\int_Q X_\nu d\nu\right)\left(\overline{\int_S X_\nu d\nu}\right)\right|}{\left\|\int_Q X_\nu d\nu\right\|_2 \left\|\int_S X_\nu d\nu\right\|_2}, \tag{1.12}$$

$$r_c(X,s) = r_c(s) := \sup \mathcal{R}_c(Q,S), \tag{1.13}$$

where each supremum is taken over all pairs of non-empty, bounded Borel sets Q and $S \subset \mathbb{R}^d$ such that

$$\left.\begin{array}{c} \textit{there exists } u \in \{1,2,\ldots,d\} \textit{ such that} \\ Q \subset \{(k_1,k_2,\ldots,k_d) \in \mathbb{R}^d : k_u \leq 0\} \\ S \subset \{(k_1,k_2,\ldots,k_d) \in \mathbb{R}^d : k_u \geq s\}. \end{array}\right\} \tag{1.14}$$

Again, for each $s \in \mathbb{R}_+$, *define*

$$q_c'(X,s) = q_c'(s) \quad := \quad \sup \frac{\left| E\left(\int_Q X_\nu d\nu\right)\left(\overline{\int_S X_\nu d\nu}\right)\right|}{\left\|\int_Q X_\nu d\nu\right\|_2 \left\|\int_S X_\nu d\nu\right\|_2}, \tag{1.15}$$

$$r_c'(X,s) = r_c'(s) \quad := \quad \sup \mathcal{R}_c(Q,S), \tag{1.16}$$

$$\zeta_c(X,s) = \zeta_c(s) \quad := \quad \sup \frac{\left| E\left(\int_Q X_\nu d\nu\right)\left(\overline{\int_S X_\nu d\nu}\right)\right|}{\lambda(Q \cup S)} \tag{1.17}$$

where each supremum is taken over all pairs of non-empty, bounded Borel sets Q *and* $S \subset \mathbb{R}^d$ *such that*

$$\left.\begin{array}{c} \textit{there exists } u \in \{1,2,\ldots,d\} \textit{ and non-empty sets} \\ Q_0, S_0 \subset \mathbb{R} \textit{ with } \operatorname{dist}(Q_0,S_0) \geq s \textit{ such that} \\ Q \subset \{(k_1,k_2,\ldots,k_d) \in \mathbb{R}^d : k_u \in Q_0\} \\ S \subset \{(k_1,k_2,\ldots,k_d) \in \mathbb{R}^d : k_u \in S_0\}. \end{array}\right\} \tag{1.18}$$

Finally, for each $s \in \mathbb{R}_+$, *define*

$$q_c^*(X,s) = q_c^*(s) \quad := \quad \sup \frac{\left| E\left(\int_Q X_\nu d\nu\right)\left(\overline{\int_S X_\nu d\nu}\right)\right|}{\left\|\int_Q X_\nu d\nu\right\|_2 \left\|\int_S X_\nu d\nu\right\|_2}, \tag{1.19}$$

$$r_c^*(X,s) = r_c^*(s) \quad := \quad \sup \mathcal{R}_c(Q,S), \tag{1.20}$$

where each supremum is taken over all pairs of non-empty, bounded Borel sets Q *and* $S \subset \mathbb{R}^d$ *such that* $\operatorname{dist}(Q,S) \geq s$.

For both definitions (subscript D and c), Cauchy's inequality implies that $r(n) \leq r'(n) \leq r^*(n) \leq 1$ and $q(n) \leq q'(n) \leq q^*(n) \leq 1$. Also, it is easy to see that $q(n) \leq r(n)$, $q'(n) \leq r'(n)$, and $q^*(n) \leq r^*(n)$. It is also worth noting that $\zeta(n) \in [0,\infty]$.

Definition 1.6. *For a random field* $X := (X_k : k \in \mathbb{Z}^d)$ *and any* $\mathbf{n} := (n_1,\ldots,n_d) \in \mathbb{Z}_+^d$, *let*

$$S(X,\mathbf{n}) := \sum_k X_k$$

where the sum is taken over all $k := (k_1,k_2,\ldots,k_d) \in \mathbb{Z}^d$ *such that* $1 \leq k_i \leq n_i$ *for each* $i = 1,2,\ldots,d$. *Often times,* $\mathbf{n}$ *takes the form* $(n,n,\ldots,n)$ *for some* $n \in \mathbb{Z}_+$. *In this case, the boldface will be dropped so that* $S(X,\mathbf{n}) = S(X,n)$. *A more general sum over a finite subset* $Q \subset \mathbb{Z}^d$ *will be denoted*

$$S(X,Q) := \sum_{k \in Q} X_k.$$

DEFINITION 1.7. *For a random field* $X := (X_\nu : \nu \in \mathbb{R}^d)$ *and any* $\mathbf{a} \in \mathbb{R}_+^d$*, let*

$$I(X, \mathbf{a}) := \int_{(0,\mathbf{a})} X_\nu d\nu$$

whenever it exists, where $(0, \mathbf{a}) := \prod_{i=1}^d (0, a_i)$ *(the Cartesian product). As in the previous definition, when* $\mathbf{a} = (a, a, \ldots, a)$ *for some* $a \in \mathbb{R}_+$ *the boldface will be dropped so that* $I(X, \mathbf{a}) = I(X, a)$*. A more general integral over any bounded Borel set* $Q \subset \mathbb{R}^d$ *will be denoted*

$$I(X, Q) := \int_Q X_\nu d\nu$$

whenever it exists.

Random fields that are CCWS have many different properties as a result of satisfying certain linear dependence conditions. The following chapter will specify what conditions need to be satisfied so that the CCWS random fields have the desired properties needed for the main results of chapters three and four.

CHAPTER 2

Background Lemmas and Theorems

The results in this chapter will be referred to several times in the following chapters. There may seem to be a lack of motivation for the results to follow, but they play a significant role in the text as a whole.

The first three lemmas are taken from Bradley [**4**] and will be used to establish bounds for the spectral density function.

Lemma 2.1. *Suppose $0 \le R < 1$, and L is a positive integer. There exists a positive constant $C = C(R, L)$ depending only on R and L such that if $X := (X_k : k \in \mathbb{Z})$ is a CCWS random sequence with $r_D(1) \le R$ and $q'_D(L) \le R$, then for every positive integer n,*

$$\mathrm{E}\,|S(X,n)|^2 \ge C \cdot n \cdot \mathrm{E}\,|X_0|^2.$$

Lemma 2.2. *Suppose $0 \le R < 1$, and L is a positive integer. Let $C := C(R, L)$ be as in Lemma 2.1. If d is a positive integer and $X := (X_k : k \in \mathbb{Z}^d)$ is a CCWS random field with $r_D(1) \le R$ and $q'_D(L) \le R$, then for every positive integer n,*

$$\mathrm{E}\,|S(X,n)|^2 \ge C^d \cdot n^d \cdot \mathrm{E}\,|X_0|^2.$$

Lemma 2.3. *Suppose $0 \le R < 1$, and L is a positive integer. If d is a positive integer and $X := (X_k : k \in \mathbb{Z}^d)$ is a CCWS random field with $q'_D(L) \le R$, then for every positive integer n,*

$$\mathrm{E}\,|S(X,n)|^2 \le L^d(1+R)^d(1-R)^{-d} \cdot n^d \cdot \mathrm{E}\,|X_0|^2.$$

Lemma 2.3 can be strengthened slightly and will be restated in a way more suitable for chapters 4 through 6. It is a consequence of Lemma 1.5 in Bradley [**5**].

Lemma 2.4. *Suppose d is a positive integer. Let $\theta := \{\theta_n\}$ be a non-increasing sequence of real numbers in $[0, 1]$ where $\lim_{n\to\infty} \theta_n < 1$. Then there exists a positive number $A := A(\theta, d)$ such that if $X := (X_k : k \in \mathbb{Z}^d)$ is a CCWS random field with $q'_D(n) \le \theta_n$ for all $n \ge 1$, then for any finite* *set $Q \subset \mathbb{Z}^d$ one has that*

$$\mathrm{E}\,|S(X,Q)|^2 \leq A \cdot |Q| \cdot \mathrm{E}\,|X_0|^2.$$

The following theorem is due to Moore [**15**], and will be used in constructing counterexamples at the end of Chapter 3.

THEOREM 2.1. *Suppose m and M ($m < M$) are positive numbers and that $X := (X_k : k \in \mathbb{Z})$ is a CCWS random sequence such that X has a spectral density f on T which satisfies $m \leq f(t) \leq M$ for all $t \in T$. Then $r'_D(1) \leq 1 - m/M < 1$.*

When constructing examples of CCWS random fields, it is easier to work with their spectral densities. The next theorem is a special case of Theorem 3.1 on page 72 of Doob [**9**], and shows that any semi-definite function on T or $[-\pi, \pi]$ is the spectral density of some CCWS random sequence.

THEOREM 2.2. *Let $\gamma(\cdot)$ be a real, positive semi-definite function on $\mathbb{Z}$ such that $\gamma(k) = \gamma(-k)$ for all $k \in \mathbb{Z}$. Then there is a real, stationary Gaussian process $(X_k : k \in \mathbb{Z})$ such that $EX_k = 0$ for all $k \in \mathbb{Z}$ and $EX_jX_i = \gamma(j-i)$ for all $i, j \in \mathbb{Z}$.*

For any non-negative, integrable, symmetric function f on T ($f(e^{i\lambda}) = f(e^{-i\lambda})$), define $\gamma(k) := \int_T t^k f(t)dm(t)$. Now, Theorem 2.2 gives the existence of a real, centered, stationary Gaussian process with f as a spectral density.

LEMMA 2.5. *Let $X := (X_k : k \in \mathbb{Z}^d)$ be a CCWS random field such that $\zeta_D(n) \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} n^{-d}\,\mathrm{E}\,|S(X,n)|^2$ exists in $[0,\infty)$.*

DEFINITION 2.1. *Suppose d is a positive integer and $X := (X_k : k \in \mathbb{Z}^d)$ is a CCWS random field such that $\zeta_D(n) \to 0$ as $n \to \infty$. For each $n \in \mathbb{Z}_+$, define $F(X,n) := n^{-d}\,\mathrm{E}\,|S(X,n)|^2$ and notice that this is real and nonnegative. Now, in reference to Lemma 2.5, define $F(X) := \lim_{n\to\infty} F(X,n)$.*

LEMMA 2.6. *Suppose d is a positive integer, $\varepsilon > 0$, and that $z := \{z_n\}$ is a sequence of non-negative numbers inside $[0,\infty]$ such that $z_n \to 0$ as $n \to \infty$. Then there exists an $L := L(d,z,\varepsilon)$ such that if $X := (X_k : k \in \mathbb{Z}^d)$ is a CCWS random field with $\zeta_D(n) \leq z_n$ for all n and $\mathrm{E}\,|X_0|^2 \leq 1$, then for all $n \geq L$, $|F(X) - F(X,n)| \leq \varepsilon$.*

The existence of $F(X)$ in Lemma 2.5, and the property it has from Lemma 2.6 play critical roles in the existence of the spectral density function for X. Both lemmas are from Bradley [**5**] and will be extended to random fields of continuous index in Chapter 4.

CHAPTER 3

Convergence of Spectral Densities for a Sequence of CCWS Random Fields

A sequence of CCWS random fields of discrete index, all of which satisfy a certain weak dependence condition and have converging covariances, will have spectral densities that converge in some way to a limiting function. In Theorem 3.2 below, each of the spectral densities are continuous, and the convergence is uniform. In Theorem 3.5, the spectral densities are not necessarily continuous and bounded between two positive constants. The limiting function is also bounded between two positive constants (not necessarily continuous), and the convergence is of a weak form that will be introduced later.

The following theorem found in Bradley [**5**], extends nicely to a result for sequences of random fields.

THEOREM 3.1. *Suppose d is a positive integer, and $X := (X_k : k \in \mathbb{Z}^d)$ is a CCWS random field. Then the following two conditions are equivalent:*

A) $\zeta_D(n) \to 0$ as $n \to \infty$;

B) X has a continuous spectral density on T^d.

The next theorem looks at a sequence of random fields that all satisfy condition A of Theorem 3.1 uniformly. It evolved from both Falk and Miller, and comes from Bradley [**5**]. It was proved for general d and a stronger linear dependence condition by Miller [**13**], and for $d = 1$ and a still stronger linear dependence condition by Falk [**11**].

THEOREM 3.2. *Suppose d is a positive integer. Suppose that for each $\ell \in \mathbb{N}$, $X^{(\ell)} := (X_k^{(\ell)} : k \in \mathbb{Z}^d)$ is a CCWS random field. Suppose that*

$$Z(n) := \sup_{\ell \geq 1} \zeta_D(X^{(\ell)}, n) \to 0$$

as $n \to \infty$. *Suppose also that for each* $k \in \mathbb{Z}^d$,

$$c_k := \lim_{\ell\to\infty} EX_k^{(\ell)}\overline{X_0^{(\ell)}}$$

exists in $\mathbb{C}$.

Referring to Theorem 3.1 for each $\ell \geq 1$, *let* $f^{(\ell)} : T^d \to [0,\infty)$ *denote the unique continuous spectral density function for the CCWS random field* $X^{(\ell)}$.

Then there exists a continuous function $f : T^d \to [0,\infty)$ *such that*

$$f^{(\ell)}(t) \to f(t)$$

uniformly on T^d *as* $\ell \to \infty$. *Furthermore, for each* $k \in \mathbb{Z}^d$,

$$c_k = \int_{T^d} t^k f(t) dm_d(t)$$

The continuous spectral density obtained from Theorem 3.1 could possibly have zeroes. In 2002, other theorems of Bradley [**4**] gave necessary and sufficient conditions for a CCWS random field to have a spectral density bounded between two positive constants. These theorems will be presented next and allow spectral densities that are not necessarily continuous.

THEOREM 3.3. *Suppose* d *is a positive integer, and* $X := (X_k : k \in \mathbb{Z}^d)$ *is a non-degenerate, CCWS random field. Then the following three conditions are equivalent:*

A) X *has a spectral density* f *on* T^d *(not necessarily continuous) such that* f *is bounded between two positive constants.*

B) $r^*_D(1) < 1$.

C) $r_D(1) < 1$, *and* $\exists n \geq 1$ *such that* $r'_D(n) < 1$.

The equivalence of (A) and (B) was done for the case $d = 1$ by Moore [**15**], but extends to general d without much difficulty. The implication (B) $\Rightarrow$ (C) is trivial by the definitions (1.3), (1.6), and (1.10). The equivalence of (C) with (A) and (B) was done in Bradley [**4**]. The proof of (C) $\Rightarrow$ (A) will be given (taken from Bradley [**4**]) because the proof of Theorem 3.5 will use some of its details.

PROOF OF (C) $\Rightarrow$ (A). Assume part (C) in Theorem 3.3. For each $t \in T^d$, define the random field $X^{(t)} := (X_k^{(t)} : k \in \mathbb{Z}^d)$ by $X_k^{(t)} := t^k X_k = e^{-i\lambda\cdot k}X_k$, where $\lambda \in (-\pi,\pi]^d$ is related to t as it was prior to Definition 1.2 (recall the notation t^k). It is easy to check that the random field $X^{(t)}$ is CCWS.

For each $n \geq 1$ define the function

$$f_n(t) := n^{-d} E|S(X^{(t)}, n)|^2. \tag{3.1}$$

There exists a positive integer L and an $R \in (0,1)$ such that $r_D(X,1) \leq R$, and $r'_D(X,L) \leq R$. For each $t \in T^d$, a simple argument using (1.3) and (1.6) gives $r_D(X^{(t)},1) = r_D(X,1) \leq R$ and $r'_D(X^{(t)},L) = r'_D(X,L) \leq R$. Let C be the positive constant from Lemma 2.1 so that (3.1) and Lemma 2.2 imply $f_n(t) \geq C^d E|X_0|^2$ for all $t \in T^d$ and all $n \geq 1$. Since $E|X_0|^2 > 0$ (X is non-degenerate)

$$\theta_1 := \inf\{f_n(t) : t \in T^d, n \geq 1\} > 0. \tag{3.2}$$

Lemma 2.3 assures that $f_n(t) \leq L^d(1+R)^d(1-R)^{-d}E|X_0|^2$ and hence,

$$\theta_2 := \sup\{f_n(t) : t \in T^d, n \geq 1\} < \infty. \tag{3.3}$$

Let $L^2_{\text{real}}(T^d)$ denote the space of real valued square integrable functions on T^d. Since $L^2_{\text{real}}(T^d)$ is a Hilbert space with inner product $\langle f,g\rangle = \int_{T^d} fg\,dm_d(t)$ under the regular L^2 norm and $\sup_{n\geq 1}\|f_n\|_2 < \infty$ by (3.2) and (3.3), there exists an $f \in L^2_{\text{real}}(T^d)$ and a subsequence $\Gamma \in \mathbb{N}$ such that

$$\lim_{n\to\infty, n\in\Gamma} \langle f_n, h\rangle = \langle f, h\rangle \qquad \forall h \in L^2_{\text{real}}(T^d). \tag{3.4}$$

This is because the closed unit ball of a Hilbert space is weakly compact (Halmos [**12**], problem 17).

Condition (A) will be proved once it is shown that

$$\theta_1 \leq f(t) \leq \theta_2 \tag{3.5}$$

for almost every $t \in T^d$, and for all $k \in \mathbb{Z}^d$,

$$EX_k\overline{X_0} = \int t^k f(t) dm_d(t). \tag{3.6}$$

Choose any $\varepsilon > 0$, and suppose the set $A := \{t \in T^d : f(t) > \theta_2 + \varepsilon\}$ has non-zero measure ($m_d(A) > 0$). Note that the identity function on A (denoted I_A) is in $L^2_{\text{real}}(T^d)$. By (3.3), $\int f_n \cdot I_A \leq \theta_2 \cdot m_d(A)$ for all $n \geq 1$ (the integral is taken over T^d with respect to the probability measure m_d). This along with the fact that $\int f \cdot I_A \geq (\theta_2 + \varepsilon) \cdot m_d(A)$ contradicts (3.4). Thus, one has $m_d(A) = 0$. With ε being arbitrary, $f(t) \leq \theta_2$ a.e. Analogously, one can use the set $B := \{t \in T^d : f(t) < \theta_1 - \varepsilon\}$ to show that $f(t) \geq \theta_1$ a.e. Hence, (3.5) holds.

Fix $k \in \mathbb{Z}^d$. Using (3.4) with $h = \cos(\lambda \cdot k)$ and $h = \sin(\lambda \cdot k)$, one has that

$$\lim_{n\to\infty, n\in\Gamma} \int_{T^d} t^k f_n(t) dm_d(t) = \int_{T^d} t^k f(t) dm_d(t).$$

The proof will be complete ((3.6) will be verified) when it is shown that

$$\lim_{n\to\infty, n\in\Gamma} \int_{T^d} t^k f_n(t) dm_d(t) = EX_k\overline{X_0}. \tag{3.7}$$

Note, that for any elements $j, \ell \in \mathbb{Z}^d$, one has that

$$\begin{aligned} \int_{T^d} t^k n^{-d} EX_j^{(t)}\overline{X_\ell^{(t)}} dm_d(t) &= \int_{T^d} e^{i\lambda\cdot k} n^{-d} EX_j^{(t)}\overline{X_\ell^{(t)}} dm_d(t) \\ &= n^{-d} \int_{T^d} e^{i\lambda\cdot k} e^{-i\lambda\cdot(j-\ell)} EX_j\overline{X_\ell} dm_d(t) \\ &= \begin{cases} n^{-d} EX_k\overline{X_0} & \text{if } j-\ell = k \\ 0 & \text{if } j-\ell \neq k. \end{cases} \end{aligned}$$

Thus, using (3.1), for any given $n \geq 1$,

$$\int t^k f_n(t) dm_d(t) = (\text{card}\, H_{k,n}) \cdot n^{-d} \cdot EX_k\overline{X_0}$$

where $H_{n,k}$ is the set defined in section 1 of Appendix A. By Lemma A.1, (3.7) holds and therefore the proof of (C) $\Rightarrow$ (A) is complete. $\square$

THEOREM 3.4. *Suppose d is a positive integer, and $X := (X_k : k \in \mathbb{Z}^d)$ is a non-degenerate CCWS random field. Then the following four conditions are equivalent:*

A) X has a positive and bounded continuous spectral density function on T^d.

B) $r_D^*(1) < 1$, and $r_D^*(n) \to 0$ as $n \to \infty$.

C) $r_D(1) < 1$, and $r'_D(n) \to 0$ as $n \to \infty$.

D) $\zeta_D(n) \to 0$ as $n \to \infty$, $r_D(1) < 1$, and $\exists n \geq 1$ such that $r'_D(n) < 1$.

The equivalence of (A), (B), and (C) was shown in Bradley [**2**] and Bradley and Utev [**7**]. The equivalence (D) $\Leftrightarrow$ (A) is given by both Theorems 3.1 and 3.3, and was pointed out by Bradley [**4**].

As Theorem 3.2 is a natural extension of Theorem 3.1, the next theorem does the same for Theorem 3.3, and is taken from Shaw [**19**]. In this theorem, the convergence of the spectral densities is not uniform, pointwise, nor in L^2. The proof will be given in detail. There will be two examples given at the end of the chapter that will show the convergence given seems to be the best one can get.

THEOREM 3.5. *Suppose d is a positive integer, and that for each $\ell \in \{1, 2, 3, \ldots\}$, $X^{(\ell)} := (X_k^{(\ell)} : k \in \mathbb{Z}^d)$ is a CCWS random field. Let $\langle f, g\rangle := \int_{T^d} f\bar{g}dm_d(t)$ denote the inner product on the complex Hilbert space $L^2(T^d)$. Suppose also that the following three conditions hold:*

i) $\sup_{\ell\geq 1} r_D(X^{(\ell)}, 1) < 1$,

ii) $\exists n \geq 1$ *such that* $\sup_{\ell\geq 1} r'_D(X^{(\ell)}, n) < 1$,

iii) $\forall k \in \mathbb{Z}^d$ $c_k := \lim_{\ell\to\infty} \mathrm{E}\, X_k^{(\ell)}\overline{X_0^{(\ell)}}$ *exists in* $\mathbb{C}$, *and* $c_0 > 0$.

Then, for each $\ell \geq 1$ sufficiently large, $X^{(\ell)}$ has a spectral density $f^{(\ell)}$ on T^d that is bounded between two positive constants. Furthermore, there exists an $f \in L^2(T^d)$ such that the following holds:

A) $\lim_{\ell\to\infty} \langle f^{(\ell)}, g\rangle = \langle f, g\rangle \quad \forall g \in L^2(T^d)$,

B) $\forall k \in \mathbb{Z}^d$, $c_k = \int_{T^d} t^k f(t)dm_d(t)$,

C) There exists θ_1, θ_2 such that for almost every $t \in T^d$, $0 < \theta_1 \leq f(t) \leq \theta_2 < \infty$, and for all $\ell \geq 1$ sufficiently large, $\theta_1 \leq f^{(\ell)}(t) \leq \theta_2$.

PROOF. Note that condition iii) allows a finite number of ℓ's in which $E|X_0^{(\ell)}|^2 = 0$ ($X^{(\ell)}$ is degenerate). Since one can always go far enough out in the sequence, assume without loss of generality that for each ℓ, the random field $X^{(\ell)}$ is non-degenerate. Under this assumption one can strengthen the conclusion by deleting "sufficiently large" following condition iii) and in part C). Also, note that condition iii) allows a finite number of ℓ's for which $E|X_0^{(\ell)}|^2 < c_0/2$ or $E|X_0^{(\ell)}|^2 > (3c_0)/2$. Again, since one can always go far enough out in the sequence, assume without loss of generality that for each $\ell \geq 1$, $c_0/2 < E|X_0^{(\ell)}|^2 < (3c_0)/2$. The rest of the proof will be under this assumption.

By conditions i) and ii) there exists an $R \in [0, 1)$ and a positive integer L such that

$$\sup_{\ell\geq 1} r_D(X^{(\ell)}, 1) \leq R; \tag{3.8}$$

$$\sup_{\ell\geq 1} r'_D(X^{(\ell)}, L) \leq R. \tag{3.9}$$

Since 3.3C) implies 3.3A), (3.8) and (3.9) imply that for every $\ell \geq 1$ the random field $X^{(\ell)}$ has a spectral density $f^{(\ell)}$ on T^d (not necessarily continuous) that is bounded between two positive constants. By a familiar fact in analysis, this spectral density function is unique disregarding sets of measure zero.

For every $\ell \geq 1$ and $t \in T^d$, define the CCWS random field $X^{(\ell,t)} := (X_k^{(\ell,t)} : k \in \mathbb{Z}^d)$ where $X_k^{(\ell,t)} := t^k X_k^{(\ell)} = e^{i\,\lambda\cdot k} X_k^{(\ell)}$. For each $\ell \in \mathbb{N}$, the proof of Theorem 3.3 C) $\Rightarrow$ A) produces the spectral density function $f^{(\ell)}$ for the random field $X^{(\ell)}$. It satisfies $\theta_1^{(\ell)} \leq f^{(\ell)}(t) \leq \theta_2^{(\ell)}$ for each

$\ell \in \mathbb{N}$ and almost every $t \in T^d$, where

$$\theta_1^{(\ell)} = \inf\{n^{-d}E|S(X^{(\ell,t)},n)|^2 : t \in T^d, n \geq 1\},$$
$$\theta_2^{(\ell)} = \sup\{n^{-d}E|S(X^{(\ell,t)},n)|^2 : t \in T^d, n \geq 1\}.$$

By (1.3), (1.6), and the definition of $X^{(\ell,t)}$, it follows that for any $\ell \geq 1$ and almost every $t \in T^d$, $r_D(X^{(\ell,t)},n) = r_D(X^{(\ell)},n)$ and $r'_D(X^{(\ell,t)},n) = r'_D(X^{(\ell)},n)$ for all $n \geq 1$. Also, note that $E|X_0^{(\ell,t)}|^2 = E|X_0^{(\ell)}|^2$ for all $\ell \geq 1$ and almost every $t \in T^d$. With these observations, (3.8), (3.9), and Lemma 2.2 give

$$E|S(X^{(\ell,t)},n)|^2 \geq C^d n^d E|X_0^{(\ell,t)}|^2 \geq C^d n^d \cdot \frac{c_0}{2}, \tag{3.10}$$

for all $\ell \geq 1$, almost every $t \in T^d$, and all $n \geq 1$ (recall $C := C(R,L)$ from Lemma 2.2). Similarly, using Lemma 2.3 instead, one has

$$E|S(X^{(\ell,t)},n)|^2 \leq L^d \frac{(1+R)^d}{(1-R)^d} n^d E|X_0^{(\ell,t)}|^2 \leq L^d \frac{(1+R)^d}{(1-R)^d} n^d \cdot \frac{3c_0}{2}, \tag{3.11}$$

for all $\ell \geq 1$, almost every $t \in T^d$, and all $n \geq 1$.

Define the constants $\theta_1 := C^d \cdot c_0/2$ and $\theta_2 := L^d[(1+R)^d/(1-R)^d] \cdot 3c_0/2$. By condition iii), (3.10), and (3.11), $0 < \theta_1 \leq n^{-d}E|S_n(X^{(\ell,t)})|^2 \leq \theta_2 < \infty$ for all $\ell \geq 1$, almost every $t \in T^d$, and for all $n \geq 1$. Hence, by the definition of $\theta_1^{(\ell)}$ and $\theta_2^{(\ell)}$,

$$\theta_1 \leq f^{(\ell)}(t) \leq \theta_2 \tag{3.12}$$

for every $\ell \geq 1$ and for almost every $t \in T^d$.

Using (3.12), and an argument similar to the one that produced (3.4) in the proof of C) $\Rightarrow$ A) in Theorem 3.3, there exists a real function $f \in L^2(T^d)$ and an infinite subsequence $\Gamma \subset \mathbb{N}$ such that

$$\forall g \in L^2(T^2), \qquad \lim_{\ell \to \infty, \ell \in \Gamma} \left\langle f^{(\ell)}, g \right\rangle = \langle f, g \rangle . \tag{3.13}$$

Again, using (3.12), and the argument for (3.5) in Theorem 3.3, one obtains

$$\theta_1 \leq f(t) \leq \theta_2 \text{ for a.e } t \in T^d, \tag{3.14}$$

and thus, part (C) holds.

Referring to iii) and (3.13), note that for $g = t^{-k} = e^{-i\lambda\cdot k}$, $\left\langle f^{(\ell)}, g\right\rangle \to \langle f, g\rangle$ as $\ell \to \infty$ along the entire sequence. Therefore,

$$\begin{aligned} \forall k \in \mathbb{Z}^d \qquad c_k = \lim_{\ell\to\infty} E X_k^{(\ell)} \overline{X_0^{(\ell)}} &= \lim_{\ell\to\infty} \int_{T^d} t^k f^{(\ell)}(t) dm_d(t) \\ &= \lim_{\ell\to\infty} \left\langle f^{(\ell)}, g\right\rangle \\ &= \langle f, g\rangle \\ &= \int_{T^d} t^k f(t) dm_d(t). \end{aligned}$$

Hence, part (B) holds. Once it is shown that (3.13) holds along the entire sequence (not just $\ell \in \Gamma$) for all $g \in L^2(T^d)$, part (A) will be shown and the proof will be complete.

Let $\mathcal{A}$ be the set of all complex functions on T^d of the form $g(t) = \sum_{k\in S} a_k t^k$ where S is a nonempty finite subset of $\mathbb{Z}^d$ and $a_k \in \mathbb{C}$ for all $k \in \mathbb{Z}^d$. By linearity,

$$\forall g \in \mathcal{A}, \qquad \lim_{\ell\to\infty} \left\langle f^{(\ell)}, g\right\rangle = \langle f, g\rangle . \tag{3.15}$$

Now, it will be shown that $\mathcal{A}$ is dense in $L^2(T^d)$. For any $f, g \in \mathcal{A}$ and any $a \in \mathbb{C}$, it is elementary to see that $f + g \in \mathcal{A}$, $fg \in \mathcal{A}$, and $af \in \mathcal{A}$, which makes $\mathcal{A}$ an algebra. Since the identity function is in $\mathcal{A}$, $\mathcal{A}$ separates points and vanishes nowhere. One can also see that if $f \in \mathcal{A}$ then $\overline{f} \in \mathcal{A}$ (self adjoint).

Since T^d is compact, the uniform closure of $\mathcal{A}$ consists of all complex continuous functions on T^d by the generalized Stone-Weierstrass theorem. Theorem 3.14 in [**16**], states that the continuous functions on T^d form a dense subset of $L^2(T^d)$. Choose any arbitrary $g \in L^2(T^d)$ and any $\varepsilon > 0$. Then there exists a continuous function g_0 such that $\|g_0 - g\|_2 < \varepsilon/2$. Since g_0 is in the uniform closure of $\mathcal{A}$ (Stone-Weierstrass), there is an $h \in \mathcal{A}$ such that $\|h - g_0\|_\infty < \varepsilon/2$. Since $m_d(T^d) = 1$, $\|h - g_0\|_2 < \varepsilon/2$. Hence, $\|h - g\|_2 \le \|h - g_0\|_2 + \|g_0 - g\|_2 < \varepsilon$. It follows that $\mathcal{A}$ is dense in $L^2(T^d)$ under the L^2 norm.

To show part (A), it suffices to show that for any fixed $g \in L^2(T^d)$ and $\varepsilon > 0$,

$$\exists L \ge 1 \text{ such that } \forall \ell \ge L, \qquad \left| \int_{T^d} (f^{(l)} - f)\bar{g}\, dm_d \right| \le \varepsilon. \tag{3.16}$$

Fix any $g \in L^2(T^d)$ and $\varepsilon > 0$. Now, find an $h \in \mathcal{A}$ so that

$$\|h - g\|_2 \le \varepsilon/(4\theta_2). \tag{3.17}$$

Using (3.15), find an $L > 0$ large enough so that

$$\forall \ell \geq L, \qquad \left| \int_{T^d} (f^{(\ell)} - f)\overline{h} dm_d \right| < \varepsilon/2. \tag{3.18}$$

Minkowski's inequality with (3.12) and (3.14) gives $\|f^{(\ell)} - f\|_2 \leq 2\theta_2$. Using Cauchy's inequality along with (3.17) and (3.18), one has that for all $\ell \geq L$,

$$\begin{aligned} \left| \int_{T^d} (f^{(\ell)} - f)\bar{g} dm_d \right| &\leq \left| \int_{T^d} (f^{(\ell)} - f)\overline{h} dm_d \right| + \left| \int_{T^d} (f^{(\ell)} - f)(\overline{g-h}) dm_d \right| \\ &< \frac{\varepsilon}{2} + \|f^{(\ell)} - f\|_2 \cdot \|g - h\|_2 \\ &\leq \frac{\varepsilon}{2} + 2\theta_2 \cdot \frac{\varepsilon}{4\theta_2} = \varepsilon. \end{aligned}$$

Thus, (3.16) holds, which was sufficient for part (A). The proof of Theorem 3.5 is complete. $\square$

COROLLARY 3.1. *Suppose d is a positive integer, and that for each $\ell \in \mathbb{N}$, $X^{(\ell)} := (X_k^{(\ell)} : k \in \mathbb{Z}^d)$ is a CCWS random field. Suppose that, along with conditions i)-iii) in Theorem 3.5, one also has that*

$$Z(n) := \sup_{\ell \geq 1} \zeta(X^{(\ell)}, n) \to 0$$

as $n \to \infty$. For each $\ell \geq 1$, let $f^{(\ell)}$ be the unique continuous spectral density function for $X^{(\ell)}$ (by Theorem 3.1). Then there exists a continuous function f on T^d such that $f^{(\ell)} \to f$ uniformly on T^d as $\ell \to \infty$. Furthermore, properties (A)-(C) of Theorem 3.5 hold.

The result follows immediately from Theorems 3.2 and 3.5.

When first looking at property (A) in Theorem 3.5, one might wonder if pointwise or L^2 convergence might hold. The following two examples show that neither of these types of convergence will hold in general. The examples are both in one dimension.

Fix m and M such that $0 < m < M < \infty$. Let $A = (M + m)/2$ and $B = (M - m)/2$. This A and B will be used in both examples. The first example will involve a sequence of continuous spectral densities, and then the second will involve a non-continuous case.

For each $\ell = 1, 2, 3, \ldots$ define the function $f^{(\ell)}(t) = B\Re(t^\ell) + A$ where $\Re(\cdot)$ denotes the real part. To simplify notation, these examples will be defined on $(-\pi, \pi]$ instead of T and integrated over $(2\pi)^{-1} d\lambda$ instead of $dm(t)$. So, for each $\ell \geq 1$, define $\tilde{f}^{(\ell)}(\lambda) := B\cos(\ell\lambda) + A$, and notice that $f^{(\ell)}(t) = \tilde{f}^{(\ell)}(\lambda)$. These are spectral densities for some real, stationary Gaussian sequence $X^{(\ell)}$ by Theorem 2.2. Trivially, one has $m \leq \tilde{f}^{(\ell)}(\lambda) \leq M$ for every ℓ and for all $\lambda \in (-\pi, \pi]$. Hence, by Theorem 2.1, $r'_D(X^{(\ell)}, 1) \leq 1 - (m/M) < 1$ for every ℓ. Since $r_D(X^{(\ell)}, 1) \leq r'_D(X^{(\ell)}, 1)$, it is easy to see that conditions i) and ii) of Theorem 3.5 are satisfied. It remains to show that condition iii) is satisfied. Note that $\mathrm{E}\, X_k^{(\ell)} \overline{X_0^{(\ell)}} = \int_T t^k f^{(\ell)}(t) dm(t) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{i\,\lambda\,k} \tilde{f}^{(\ell)}(\lambda) d\,\lambda$. For fixed k,

$$
\begin{aligned}
c_k &= \lim_{\ell\to\infty} \mathrm{E}\, X_k^{(\ell)} \overline{X_0^{(\ell)}} \\
&= \lim_{\ell\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\,\lambda\,k} \cdot [B\cos(\ell\,\lambda) + A] d\,\lambda \\
&= \lim_{\ell\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} B e^{i\,\lambda\,k} \cos(\ell\,\lambda) d\,\lambda + \frac{1}{2\pi} \int_{-\pi}^{\pi} A e^{i\,\lambda\,k} d\,\lambda \\
&= \begin{cases} A & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \neq 0 \end{cases} .
\end{aligned} \tag{3.19}
$$

The first integral in the third line of (3.19) vanishes when ℓ gets larger than $|k|$. Hence, part iii) is satisfied and the theorem holds. Referring to (3.19) and part (B) in the conclusion of Theorem 3.5, the limit function is $f = A$. For any ℓ,

$$
\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \tilde{f}^{(\ell)}(\lambda) - A \right|^2 d\,\lambda &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |B\cos(\ell\,\lambda)|^2 \, d\,\lambda \\
&= \frac{B^2}{2\pi} \int_{-\pi}^{\pi} \cos^2(\ell\,\lambda) d\,\lambda \\
&= \frac{B^2}{2}.
\end{aligned}
$$

Hence, $\tilde{f}^{(\ell)}$ does not converge to $f = A$ in L^2. It is also easy to see that $\tilde{f}^{(\ell)}$ does not converge to f pointwise almost everywhere.

The second example will use the Rademacher functions. A definition of these can be found at the bottom of page 5 in Billingsley [**1**]. The first two are defined as follows:

$$
r_1(x) = \begin{cases} -1 & \text{for} \quad x \in [0, \frac{1}{2}] \\ 1 & \text{for} \quad x \in \left(\frac{1}{2}, 1\right] \end{cases} \qquad r_2(x) = \begin{cases} -1 & \text{for} \quad x \in \left[0, \frac{1}{4}\right] \cup \left(\frac{1}{2}, \frac{3}{4}\right] \\ 1 & \text{for} \quad x \in \left(\frac{1}{4}, \frac{1}{2}\right] \cup \left(\frac{3}{4}, 1\right] \end{cases} .
$$

This is the usual notation for Rademacher functions and should not be confused with linear dependence coefficients r_D, r_c, r'_D, and r'_c. This example requires a rescaling and a reflection of these functions. Define

$$
r'_1(\lambda) = \begin{cases} r_1\left(\frac{\lambda}{\pi}\right) & \lambda \in [0, \pi] \\ r_1\left(-\frac{\lambda}{\pi}\right) & \lambda \in [-\pi, 0) \end{cases} \qquad r'_2(\lambda) = \begin{cases} r_2\left(\frac{\lambda}{\pi}\right) & \lambda \in [0, \pi] \\ r_2\left(-\frac{\lambda}{\pi}\right) & \lambda \in [-\pi, 0) \end{cases}
$$

and the rest of the r'_i accordingly.

Once again, this example will be defined on $(-\pi,\pi]$ instead of T. For each $\ell = 1,2,3,\dots$ define the function

$$f^{(\ell)}(\lambda) = Br'_\ell(\lambda) + A.$$

Again, these functions are spectral densities for a real, stationary Guassian sequence by Theorem 2.2. Conditions i) and ii) in Theorem 3.5 are satisfied by the same argument in the first example. Now it remains to show that iii) holds, which will follow if one shows that

$$\left\langle f^{(\ell)}, e^{i\,\lambda\,k}\right\rangle \to \left\langle A, e^{i\,\lambda\,k}\right\rangle \text{ as } \ell \to \infty, \tag{3.20}$$

for each $k \in \mathbb{Z}$ and $\lambda \in (-\pi,\pi]$.

The functions r_ℓ form an orthonormal set since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} r'_i(\lambda) r'_j(\lambda) d\,\lambda = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

This follows from equation 1.17 in Billingsley [**1**] and a simple computation (because of the rescaling). These functions are contained in some maximal orthonormal set $\mathcal{M}$ by Theorem 4.22 in Rudin [**16**]. Since $t^k = e^{i\,\lambda\,k} \in L^2(T)$, one has $\|e^{i\,\lambda\,k}\|_2^2 = \sum_{u\in\mathcal{M}} |\left\langle e^{i\,\lambda\,k}, u\right\rangle|^2 < \infty$. This converging sum implies that $|\left\langle e^{i\,\lambda\,k}, u\right\rangle| \to 0$, and in particular $|\left\langle e^{i\,\lambda\,k}, r'_\ell\right\rangle| \to 0$ as $\ell \to \infty$, since $r'_\ell \in \mathcal{M}$. More conveniently, one has

$$|\left\langle r'_\ell, e^{ik\,\lambda}\right\rangle| \to 0 \text{ as } \ell \to \infty. \tag{3.21}$$

By the linearity of the inner product, $\left\langle Br'_\ell + A, e^{i\,\lambda\,k}\right\rangle = B\left\langle r'_\ell, e^{i\,\lambda\,k}\right\rangle + \left\langle A, e^{i\,\lambda\,k}\right\rangle$. From this computation and (3.21), (3.20) holds. Hence, condition iii) is satisfied and Theorem 3.5 applies. From (3.20), the limit function is again $f = A$. The $f^{(\ell)}$'s converge nowhere to A pointwise. Also, for every $\ell \geq 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f^{(\ell)}(\lambda) - A|^2 d\,\lambda = \frac{B^2}{2\pi}\int_{-\pi}^{\pi} (r'_\ell(\lambda))^2 d\,\lambda = B^2$$

and so the $f^{(\ell)}$'s do not converge to A in L^2 either.

CHAPTER 4

Random Fields of Continuous Index

For a CCWS random field $Y := (Y_k : k \in \mathbb{Z}^d)$, the condition $\zeta_D(Y, n) \to 0$ as $n \to \infty$ is sufficient for the existence of a continuous spectral density [**5**]. When the CCWS random field is of continuous index $X := (X_\nu : \nu \in \mathbb{R}^d)$, the condition $\zeta_c(X, s) \to 0$ does not seem to be sufficient for a continuous spectral density. If we integrate the X_ν over blocks (translations of $[0, 1]^d$), one can generate a discrete-indexed random field. If one works with these fields, the lemmas in Chapter 2 can be extended to include CCWS random fields indexed by $\mathbb{R}^d$. In turn, this will aid in extending Theorem 3.1 to random fields of continuous index by the addition of a few more conditions. The rest of this work will be devoted to this task.

Definition 4.1. *A CCWS random field $X := (X_\nu : \nu \in \mathbb{R}^d)$ is mean square continuous (MSC) if the complex covariance function γ is continuous at the origin.*

Suppose that X is a CCWS random field that is MSC. For a fixed $\varepsilon > 0$, let $\delta > 0$ be such that $\|\nu\| < \delta$ implies that $|\gamma(\nu) - \gamma(0)| < \varepsilon/2$. Then for all $\nu, r \in \mathbb{R}^d$,

$$\begin{aligned} \mathrm{E}\,|X_\nu - X_r|^2 &= \left(\mathrm{E}\,X_\nu\overline{X_\nu} - \mathrm{E}\,X_\nu\overline{X_r} - \mathrm{E}\,X_r\overline{X_\nu} + \mathrm{E}\,X_r\overline{X_r}\right) \\ &= \left(\gamma(0) - \gamma(\nu - r) - \gamma(r - \nu) + \gamma(0)\right). \end{aligned}$$

Thus, whenever $\|\nu - r\| < \delta$, one has that $\mathrm{E}\,|X_\nu - X_r|^2 < \varepsilon$. It will now be shown that if a CCWS random field is MSC, then the complex covariance function is uniformly continuous over $\mathbb{R}^d$.

If X is degenerate, then the covariance function is the constant function 0, and therefore, uniformly continuous. Now assume that X is non-degenerate so that $\|X_0\|_2 > 0$. Choose $\varepsilon > 0$ arbitrarily. Using the previous argument and the fact that $\|X_0\|_2 < \infty$ by weak stationarity (Definition 1.1), let $\delta > 0$ be small enough so that if $\|\nu - r\| < \delta$, then $\|X_\nu - X_r\|_2 < \varepsilon/\|X_0\|_2$.

Then, for all ν, $r \in \mathbb{R}^d$ such that $\|\nu - r\| < \delta$,

$$\begin{aligned} |\gamma(\nu) - \gamma(r)| &= |\operatorname{E} X_\nu \overline{X}_0 - \operatorname{E} X_r \overline{X}_0| \\ &= |\operatorname{E}(X_\nu - X_r)\overline{X}_0| \\ &\le \|X_\nu - X_r\|_2 \cdot \|X_0\|_2 \\ &< \varepsilon. \end{aligned}$$

Thus, the complex covariance function γ is uniformly continuous on $\mathbb{R}^d$.

Under the assumption that $(\nu, \omega) \mapsto X_\nu(\omega)$ is measurable with respect to the product σ-field $\mathcal{R}^d \times \mathcal{F}$ (which is understood in this text), one already has mean square continuity from lines 13-16 on page 60, and lines 10-12 of section 3 on page 518 of [**9**].

THEOREM 4.1. *Let $X := (X_\nu : \nu \in \mathbb{R}^d)$ be a non-degenerate, CCWS random field. Suppose that $\zeta_c(s) \to 0$ as $s \to \infty$, and $r_c'(A) < 1$ for some $A > 0$. If the function $T : \mathbb{R}^d \to \mathbb{R}$ defined by*

$$T(x) := \operatorname{E}\left| \int_{[0,1]^d} e^{-ix\cdot\nu} X_\nu \, d\nu \right|^2$$

is integrable, then X has a nonnegative, continuous spectral density function on $\mathbb{R}^d$.

Curtis Miller posed a theorem similar to this using the stronger hypothesis $\rho^*(s) \to 0$ as $s \to \infty$ [**14**]. The ρ^* coefficient is very similar to r_D^* defined in (1.10), and a definition can be found in either Bradley [**3**] or Miller [**13**]. In Miller's statement of the theorem, he defines the function $T(x)$ by

$$T(x) := \int_{[-1,1]^d} e^{-ix\cdot\nu} \left(\prod_{i=1}^d (1 - |\nu_i|) \right) \gamma(\nu) d\nu. \tag{4.1}$$

These two definitions of $T(x)$ are equal. Both will be used extensively, and a derivation of the equality can be found in Appendix A. It is done over the general interval $[0, L]^d$ for some $L > 0$.

For any $a \in \mathbb{R}_+$, let $\lfloor a \rfloor$ denote the greatest integer less than or equal to a. If $\mathbf{a} := (a_1, a_2, \ldots, a_d) \in \mathbb{R}_+^d$ and $k := (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$, then let $(\mathbf{0}, \mathbf{a}) := \prod_{i=1}^d (0, a_i)$ (Cartesian product) and $\mathbf{a}k := (a_1k_1, a_2k_2, \ldots, a_dk_d)$, i.e. coordinatewise multiplication. From this, define $(-\mathbf{a}, \mathbf{0}) + \mathbf{a}k := \prod_{i=1}^d ((k_i - 1)a_i, k_ia_i)$ (Cartesian product). The next lemma is an extension of Lemma 2.4 to random fields of continuous index. Miller first obtained this result under the hypothesis $\rho^*(n) < 1$ for some $n > 0$ [**14**].

LEMMA 4.1. *Suppose d is a positive integer. Let $\theta := \{\theta_n\}$ be a non-increasing sequence of real numbers in $[0, 1]$ where $\lim_{n\to\infty} \theta_n < 1$. Then there exists a positive number $B := B(\theta, d)$ such* *that if $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field with $q_c'(n) \le \theta_n$ for all $n \ge 1$, then for any* $\mathbf{a} := (a_1, a_2, \ldots, a_d) \in \mathbb{R}_+^d$,

$$\mathrm{E}\,|\,\mathrm{I}(X,\mathbf{a})|^2 \le B \cdot \left(\prod_{i=1}^d a_i\right) \cdot \|X_0\|_2^2.$$

Proof. Let $\theta_0 := 1$ and define the sequence $\theta' := \{\theta_n'\}$ by $\theta_n' = \theta_{\lfloor (n-1)/2 \rfloor}$. Let $A_j := A(\theta', j)$ be the constant obtained from Lemma 2.4 for each $j \in \{1, 2, \ldots, d\}$ and then let $B := \max\{1, A_1, A_2, \ldots, A_d\}$ (with a little work, one can take $B = A_d$). It will be shown that Lemma 4.1 holds with this B.

Suppose that $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field such that $q_c'(n) \le \theta_n$ for all $n \ge 1$. Fix any $\mathbf{a} \in \mathbb{R}_+^d$. For each $i = 1, 2, \ldots, d$, define $a_i' := a_i/(1 + \lfloor a_i \rfloor)$. Then $a_i' < 1$ for every $i \in \{1, 2, \ldots, d\}$. Let $\mathbf{a}' := (a_1', a_2', \ldots, a_d')$. A simple application of Hölder's inequality and Fubini's Theorem yields

$$\begin{aligned}
\text{(4.2)} \qquad \mathrm{E}\left|\int_{(\mathbf{0},\mathbf{a}')} X_\nu d\nu\right|^2 &\le \mathrm{E}\left(\int_{(\mathbf{0},\mathbf{a}')} |X_\nu| d\nu\right)^2 \\
&\le \mathrm{E}\left(\int_{(\mathbf{0},\mathbf{a}')} |X_\nu|^2 d\nu \cdot \int_{(\mathbf{0},\mathbf{a}')} 1^2 d\nu\right) \\
&= \left(\prod_{i=1}^d a_i'\right)^2 \cdot \|X_0\|_2^2.
\end{aligned}$$

From this, the proof is complete in the case when $a_i < 1$ for all $i \in \{1, 2, \ldots, d\}$ (since $a_i = a_i'$ in this case). Now assume $a_i \ge 1$ for at least one i.

Let $Q := \{i \in \{1, 2, \ldots, d\} : a_i \ge 1\}$. Without loss of generality, one can assume that $Q = \{1, \ldots, j\}$ for some $j \in \{1, 2, \ldots, d\}$ by permuting the indices if necessary. For each $k := (k_1, k_2, \ldots, k_j) \in \mathbb{Z}^j$, let $k' := (k_1, \ldots, k_j, 1, \ldots, 1) \in \mathbb{Z}^d$. With this notation in place, let $0' := (0, \ldots, 0, 1, \ldots, 1)$ where there are j 0's and $d - j$ 1's. Now, for $k \in \mathbb{Z}^j$, define $Y_k := \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'k'} X_\nu d\nu$. Then $Y := (Y_k : k \in \mathbb{Z}^j)$ is a discrete parameter random field. Since X is complex and centered, Y is complex and Fubini gives the fact that Y is centered. The weak stationarity of Y will be obtained by using the weak stationarity of X and a few applications of Fubini's theorem. For h, $k \in \mathbb{Z}^j$ (let

h' be defined as k' is above and note that $h'-k' = (h-k)' - 0'$),

$$\begin{aligned}
\mathrm{E}\, Y_h \overline{Y}_k &= \mathrm{E}\left(\int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'h'} X_\nu d\nu \cdot \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'k'} \overline{X}_\xi d\xi\right) \\
&= \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'h'} \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'k'} \mathrm{E}\left(X_\nu \overline{X}_\xi\right) d\xi d\nu \\
&= \int_{(-\mathbf{a}',\mathbf{0})} \int_{(-\mathbf{a}',\mathbf{0})} \mathrm{E}\left(X_{\nu+\mathbf{a}'h'} \overline{X}_{\xi+\mathbf{a}'k'}\right) d\xi d\nu \\
&= \int_{(-\mathbf{a}',\mathbf{0})} \int_{(-\mathbf{a}',\mathbf{0})} \mathrm{E}\left(X_{\nu+\mathbf{a}'(h'-k')} \overline{X}_{\xi}\right) d\xi d\nu \\
&= \int_{(-\mathbf{a}',\mathbf{0})} \int_{(-\mathbf{a}',\mathbf{0})} \mathrm{E}\left(X_{\nu+\mathbf{a}'[(h-k)'-0']} \overline{X}_{\xi}\right) d\xi d\nu \\
&= \int_{(-\mathbf{a}',\mathbf{0})} \int_{(-\mathbf{a}',\mathbf{0})} \mathrm{E}\left(X_{\nu+\mathbf{a}'(h-k)'} \overline{X}_{\xi+\mathbf{a}'0'}\right) d\xi d\nu \\
&= \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'(h-k)'} \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'0'} \mathrm{E}\left(X_\nu \overline{X}_\xi\right) d\xi d\nu \\
&= \mathrm{E}\left(\int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'(h-k)'} X_\nu d\nu \cdot \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'0'} \overline{X}_\xi d\xi\right) \\
&= \mathrm{E}\, Y_{h-k} \overline{Y}_0,
\end{aligned}$$

and hence, Y is weakly stationary. Since Y is weakly stationary,

$$\|Y_0\|_2^2 = \|Y_1\|_2^2 = \mathrm{E}\left|\int_{(0,\mathbf{a}')} X_\nu d\nu\right|^2,$$

(Remark 1.1) and therefore (4.2) yields

$$\|Y_0\|_2^2 \le \left(\prod_{i=1}^{d} a_i'\right)^2 \cdot \|X_0\|_2^2. \tag{4.3}$$

Notice that with the way Y is defined, (1.5) and the fact that $1/2 \le a_i' < 1$ for each $i \in Q$ gives $q_D'(Y,n) \le q_c'(X,(n-1)/2)$ for all $n \ge 2$. Since $q_c'(X,(n-1)/2) \le \theta_{\lfloor (n-1)/2 \rfloor}$ for all $n \ge 2$, $q_D'(Y,n) \le \theta_n'$ for all $n \ge 1$. Let $\lfloor \tilde{\mathbf{a}} \rfloor + \mathbf{1} := (\lfloor a_1 \rfloor + 1, \lfloor a_2 \rfloor + 1, \ldots, \lfloor a_j \rfloor + 1)$, and recall that

$a_i = a_i'(\lfloor a_i \rfloor + 1)$. Apply Lemma 2.4 and (4.3) above to get

$$\begin{aligned}
\mathrm{E}\,|\,\mathrm{I}(X,\mathbf{a})|^2 &= \mathrm{E}\,|S(Y, \lfloor \tilde{\mathbf{a}} \rfloor + \mathbf{1})|^2 \\
&\le A_j \cdot \prod_{i=1}^{j} (\lfloor a_i \rfloor + 1) \cdot \|Y_0\|_2^2 \\
&= A_j \cdot \prod_{i=1}^{d} (\lfloor a_i \rfloor + 1) \cdot \|Y_0\|_2^2 \\
&\le B \cdot \prod_{i=1}^{d} (\lfloor a_i \rfloor + 1) \cdot \left(\prod_{i=1}^{d} a_i' \right)^2 \cdot \|X_0\|_2^2 \\
&= B \cdot \left(\prod_{i=1}^{d} a_i'(\lfloor a_i \rfloor + 1) \right) \cdot \prod_{i=1}^{d} a_i' \cdot \|X_0\|_2^2 \\
&= B \cdot \left(\prod_{i=1}^{d} a_i \right) \cdot \prod_{i=1}^{d} a_i' \cdot \|X_0\|_2^2 \\
&\le B \cdot \left(\prod_{i=1}^{d} a_i \right) \cdot \|X_0\|_2^2.
\end{aligned}$$

Thus, the proof of Lemma 4.1 is complete. □

A slightly modified version of the discrete-indexed random field Y in the proof of Lemma 4.1 plays a significant role in the rest of this chapter and Chapter 5. Define $Y := (Y_k : k \in \mathbb{Z}^d)$ by

$$Y_k = \int_{(-1,0)^d + k} X_\nu \, d\nu. \tag{4.4}$$

By a calculation in the proof of Lemma 4.1 (with a' replaced by $(1, 1, \dots, 1)$), one has that Y is a CCWS random field. With the definition in (4.4), notice that $S(Y, \mathbf{n}) = \mathrm{I}(X, \mathbf{n})$ and $\zeta_D(Y, n) \le \zeta_c(X, n-1)$ for $n \ge 2$ (recall both (1.7) and (1.17)). If one were to assume that $\zeta_c(X, n) \to 0$, these properties and Lemma 2.5 imply that $\lim_{a\to\infty} \lfloor a \rfloor^{-d}\, \mathrm{E}\,|\,\mathrm{I}(X, \lfloor a \rfloor)|^2$ exists (recall that $\mathrm{I}(X, a) = \mathrm{I}(X, \mathbf{a})$ when $\mathbf{a} = (a, a, \dots, a)$). One would like this limit to hold for $a^{-d}\, \mathrm{E}\,|\,\mathrm{I}(X, a)|^2$, so it needs to be shown that

$$|a^{-d}\, \mathrm{E}\,|\,\mathrm{I}(X, a)|^2 - \lfloor a \rfloor^{-d}\, \mathrm{E}\,|\,\mathrm{I}(X, \lfloor a \rfloor)|^2| \to 0 \text{ as } a \to \infty. \tag{4.5}$$

The following lemma was first proved by Curtis Miller [**14**] under the stronger hypothesis that $\rho^*(n) < 1$ for some $n \ge 1$ and will help obtain (4.5).

LEMMA 4.2. *Suppose that d is a positive integer, $\theta := \{\theta_n\}$ is a non-increasing sequence of numbers in $[0, 1]$ such that $\lim_{n\to\infty} \theta_n < 1$, and $B := B(\theta, d)$ is the constant from Lemma 4.1. If*

$X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field with $q_c'(n) \le \theta_n$ for all $n \ge 1$, then for any $a \in \mathbb{R}_+$ one has that

$$\left| \mathrm{E}\,|\,\mathrm{I}(X,a)|^2 - \mathrm{E}\,|\,\mathrm{I}(X,\lfloor a \rfloor)|^2 \right| \le 2da^{d-1/2}B\|X_0\|_2^2. \tag{4.6}$$

Proof. The proof will be two calculations. The first one uses Hölder's inequality and Lemma 4.1, and is given by

$$\begin{aligned}
&\|\mathrm{I}(X,a) - \mathrm{I}(X,\lfloor a \rfloor)\|_2 \\
&= \left\| \int_{(0,a)^d} X_\nu d\nu - \int_{(0,\lfloor a \rfloor)^d} X_\nu d\nu \right\|_2 \\
&= \left\| \int_{(0,a)} \cdots \int_{(0,a)} X_\nu d\nu - \int_{(0,\lfloor a \rfloor)} \cdots \int_{(0,\lfloor a \rfloor)} X_\nu d\nu \right\|_2 \\
&= \Bigg\| \sum_{k=1}^d \Bigg(\int_{(0,a)} \cdots \int_{(0,a)} \left(\int_{(0,\lfloor a \rfloor)} \cdots \int_{(0,\lfloor a \rfloor)} X_\nu d\nu_1 \dots d\nu_{k-1} \right) d\nu_k \dots d\nu_d \\
&\qquad - \int_{(0,a)} \cdots \int_{(0,a)} \left(\int_{(0,\lfloor a \rfloor)} \cdots \int_{(0,\lfloor a \rfloor)} X_\nu d\nu_1 \dots d\nu_k \right) d\nu_{k+1} \dots d\nu_d \Bigg) \Bigg\|_2 \\
&= \left\| \sum_{k=1}^d \int_{(0,a)} \cdots \int_{(0,a)} \int_{(\lfloor a \rfloor,a)} \int_{(0,\lfloor a \rfloor)} \cdots \int_{(0,\lfloor a \rfloor)} X_\nu d\nu_1 \dots d\nu_{k-1} d\nu_k d\nu_{k+1} \dots d\nu_d \right\|_2 \\
&\le \sum_{k=1}^d \left\| \int_{(0,a)} \cdots \int_{(0,a)} \int_{(\lfloor a \rfloor,a)} \int_{(0,\lfloor a \rfloor)} \cdots \int_{(0,\lfloor a \rfloor)} X_\nu d\nu_1 \dots d\nu_{k-1} d\nu_k d\nu_{k+1} \dots d\nu_d \right\|_2 \\
&\le \sum_{k=1}^d (a^{d-k}(a - \lfloor a \rfloor)\lfloor a \rfloor^{k-1} B \|X_0\|_2^2)^{1/2} \\
&\le da^{(d-1)/2} B^{1/2} \|X_0\|_2.
\end{aligned}$$

The second and final calculation uses the reverse triangle inequality, the first calculation done above, and Lemma 4.1. It is given by

$$\begin{aligned}
&\big|\mathrm{E}\,|\,\mathrm{I}(X,a)|^2 - \mathrm{E}\,|\,\mathrm{I}(X,\lfloor a\rfloor)|^2\big| \\
&\quad = \big|\|\,\mathrm{I}(X,a)\|_2^2 - \|\,\mathrm{I}(X,\lfloor a\rfloor)\|_2^2\big| \\
&\quad = |\|\,\mathrm{I}(X,a)\|_2 - \|\,\mathrm{I}(X,\lfloor a\rfloor)\|_2| \cdot |\|\,\mathrm{I}(X,a)\|_2 + \|\,\mathrm{I}(X,\lfloor a\rfloor)\|_2| \\
&\quad \le \|\,\mathrm{I}(X,a) - \mathrm{I}(X,\lfloor a\rfloor)\|_2 \cdot 2a^{d/2}B^{1/2}\|X_0\|_2 \\
&\quad \le 2da^{d-1/2}B\|X_0\|_2^2.
\end{aligned}$$

Thus, (4.6) holds and the proof is complete. □

LEMMA 4.3. *Suppose that $\theta := \{\theta_n\}$ and $z := \{z_n\}$ are non-increasing sequences in $[0,1]$ and $[0,\infty]$ respectively such that $\lim_{n\to\infty}\theta_n < 1$ and $\lim_{n\to\infty} z_n = 0$. If $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field with $q_c'(n) \le \theta_n$ and $\zeta_c(n) \le z_n$ for all $n \ge 1$, then $\lim_{a\to\infty} a^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X,a)|^2$ exists in $[0,\infty)$.*

PROOF. The proof is trivial in the degenerate case, so assume that $\|X_0\|_2^2 > 0$. Let $B := B(\theta, d)$ be the constant from Lemma 4.1. Use Lemma 4.2 and divide both sides of (4.6) by a^d to get

$$\big|a^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X,a)|^2 - a^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X,\lfloor a\rfloor)|^2\big| \le \frac{2dB\|X_0\|_2^2}{a^{1/2}}. \tag{4.7}$$

Use Lemma 4.1 to get that

$$\begin{aligned}
\big|a^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X,\lfloor a\rfloor)|^2 - \lfloor a\rfloor^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X,\lfloor a\rfloor)|^2\big| &= \lfloor a\rfloor^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X,\lfloor a\rfloor)|^2 \left|\frac{\lfloor a\rfloor^d}{a^d} - 1\right| \\
&\le \left|\frac{\lfloor a\rfloor^d}{a^d} - 1\right| \cdot B\|X_0\|_2^2.
\end{aligned} \tag{4.8}$$

For any $\varepsilon > 0$, one can find an $L > 0$ large enough so that for any $a \ge L$, both of the following hold:

$$\frac{2dB\|X_0\|_2^2}{a^{1/2}} < \frac{\varepsilon}{2}, \tag{4.9}$$

$$\left|\frac{\lfloor a\rfloor^d}{a^d} - 1\right| < \frac{\varepsilon}{2B\|X_0\|_2^2}. \tag{4.10}$$

Using (4.7), (4.8), (4.9), and (4.10) with the triangle inequality, one has

$$|a^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X,a)|^2 - \lfloor a\rfloor^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X,\lfloor a\rfloor)|^2| < \varepsilon \tag{4.11}$$

for any $a \ge L$, which confirms (4.5) since ε is arbitrary. Let Y be the CCWS random field defined by (4.4). Since $\zeta_D(Y,n) \le \zeta_c(X,n-1)$ for $n \ge 2$, one has that $\zeta_D(Y,n) \le z_{n-1}$ for all $n \ge 2$. Lemma

2.5 implies that $\lim_{a\to\infty} \lfloor a \rfloor^{-d} \mathrm{E}\,|\,\mathrm{I}(X, \lfloor a \rfloor)|^2$ exists in $[0, \infty)$, since $z_n \to 0$ and $S(Y, \lfloor a \rfloor) = I(X, \lfloor a \rfloor)$. This and (4.5) imply that $\lim_{a\to\infty} a^{-d} \mathrm{E}\,|\,\mathrm{I}(X, a)|^2$ exists, and therefore the proof is complete. □

Now, Lemma 2.6 will be extended to random fields of continuous index with an added condition. By Definition 2.1, $F(Y, n) = n^{-d} \mathrm{E}\,|S(Y, n)|^2$ for a discrete-indexed random field Y. For a continuous-indexed random field X, it will be understood that $F(X, a) := a^{-d} \mathrm{E}\,|\,\mathrm{I}(X, a)|^2$. Also, with reference to Lemma 4.3, let $F(X) := \lim_{a\to\infty} F(X, a)$ whenever the limit exists.

LEMMA 4.4. *Suppose that $\theta := \{\theta_n\}$ and $z := \{z_n\}$ are non-increasing sequences in $[0, 1]$ and $[0, \infty]$ respectively such that $\lim_{n\to\infty} \theta_n < 1$ and $\lim_{n\to\infty} z_n = 0$. Then for any given $\varepsilon > 0$, there exists an $L := L(\varepsilon, \theta, z) > 0$ such that if $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field with $\mathrm{E}\,|X_0|^2 \le 1$, $q_c'(n) \le \theta_n$ and $\zeta_c(n) \le z_n$ for all $n \ge 1$, then $|F(X) - F(X, a)| \le \varepsilon$ for all $a \ge L$.*

PROOF. Let $\varepsilon > 0$ be fixed and arbitrary. Define the sequence $z_n' := z_{n-1}$ with $z_0 = \infty$, and let $L_1 := L_1(z_n', \varepsilon/2)$ be the constant from Lemma 2.6. Let $B := B(\theta, d)$ be the constant from Lemma 4.1. Choose $L_2 > 0$ large enough so that for all $a \ge L_2$, both of the following hold:

$$\frac{2dB}{a^{1/2}} < \frac{\varepsilon}{4}, \tag{4.12}$$

$$\left| \frac{\lfloor a \rfloor^d}{a^d} - 1 \right| < \frac{\varepsilon}{4B}. \tag{4.13}$$

If Y is defined as it is in (4.4), then $\|Y_0\|_2^2 \le 1$ by (4.2) and the fact that $\|X_0\|_2^2 \le 1$. Since both $F(Y) = \lim_{n\to\infty} F(Y, n)$ and $F(X) = \lim_{a\to\infty} F(X, a)$ exist, they must be equal (recall $F(Y, \lfloor a \rfloor) = \lfloor a \rfloor^{-d} \mathrm{E}\,|S(Y, \lfloor a \rfloor)|^2 = \lfloor a \rfloor^{-d} \mathrm{E}\,|\,\mathrm{I}(X, \lfloor a \rfloor)|^2 = F(X, \lfloor a \rfloor)$). The definition of L_1 gives $|F(Y) - F(Y, \lfloor a \rfloor)| < \varepsilon/2$ for all $a \ge L_1$, which is the same as $|F(X) - F(X, \lfloor a \rfloor)| < \varepsilon/2$. Since $\|X_0\|_2^2 \le 1$, (4.7), (4.8), and the triangle inequality together with (4.12) and (4.13) give $|F(X, \lfloor a \rfloor) - F(X, a)| < \varepsilon/2$, for all $a \ge L_2$. If $L := \max\{L_1, L_2\}$, then the triangle inequality yields $|F(X) - F(X, a)| < \varepsilon$ for all $a \ge L$.

□

CHAPTER 5

The Random Field $X^{<x>}$

Given the random field $X := (X_\nu : \nu \in \mathbb{R}^d)$ and any $x \in \mathbb{R}^d$, define the random field

$$X^{<x>} := (X_\nu^{<x>} : \nu \in \mathbb{R}^d) \text{ where } X_\nu^{<x>} := e^{-ix\cdot\nu} X_\nu. \tag{5.1}$$

Also, define the random field

$$Y^{<x>} := (Y_k^{<x>} : k \in \mathbb{Z}^d) \text{ where } Y_k^{<x>} := \int_{(-1,0)^d+k} X_\nu^{<x>} d\nu. \tag{5.2}$$

Observe that with this definition in place (and referring to Remark 1.1), $\|Y_1^{<x>}\|_2^2 = T(x)$.

Since X is CCWS, an elementary calculation will show that the random field $X^{<x>}$ is CCWS. Following the same argument that is in the proof of Lemma 4.1, one can see that $Y^{<x>}$ is CCWS, also. The next lemma was established in the case of discrete-indexed random fields by Bradley [**5**], and had the immediate consequence $\zeta_D(Y^{<x>}, n) \le 16\zeta_D(Y, n)$. The analogous result $\zeta_c(X^{<x>}, s) \le 16\zeta_c(X, s)$ will follow from the next lemma. Then, one will have $\zeta_c(X^{<x>}, s) \to 0$ as $s \to \infty$ whenever $\zeta_c(X, s) \to 0$ as $s \to \infty$. In (1.17), the coefficient ζ_c was defined for CCWS random fields only. The same definition will be adapted verbatim for the random fields in Lemma 5.1 below.

Lemma 5.1. *Suppose d is a positive integer, and $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a centered and complex (not necessarily weakly stationary) random field with $\mathrm{E}\,|X_\nu|^2 < \infty$ for all $\nu \in \mathbb{R}^d$ and $\int_B \|X_\nu\|_2 d\nu < \infty$ for each bounded Borel set $B \subset \mathbb{R}^d$. Suppose s is a positive real number with $\zeta_c(X, s) < \infty$, and that Q and S are nonempty, disjoint, bounded Borel subsets of $\mathbb{R}^d$ satisfying (1.18).*

If $a(\cdot)$ is a Borel function on $Q \cup S$ such that $a(\nu) \in [0, 1]$ for all $\nu \in Q \cup S$, then

$$\left|\mathrm{E}\left(\int_Q a(\nu)X_\nu d\nu\right)\left(\int_S \overline{a(\nu)X_\nu} d\nu\right)\right| \le \zeta_c(X, s)\,\lambda(Q \cup S). \tag{5.3}$$

If $c(\cdot)$ is a complex valued Borel function on $Q \cup S$ with $|c(\nu)| \le 1$ for all $\nu \in Q \cup S$, then

$$\left|\mathrm{E}\left(\int_Q c(\nu)X_\nu d\nu\right)\left(\int_S \overline{c(\nu)X_\nu} d\nu\right)\right| \le 16\zeta_c(X, s)\,\lambda(Q \cup S). \tag{5.4}$$

Proof. Let $a : Q \cup S \to [0, 1]$ be an arbitrary Borel function. For each positive integer L, partition Q into $\{Q_0^{(L)}, Q_1^{(L)}, \dots, Q_L^{(L)}\}$ such that $Q_j^{(L)} := \{\nu \in Q : a(\nu) \in [j/L, (j+1)/L)\}$.

Partition S accordingly. Since $a(\nu) \in [0,1]$, $Q_L^{(L)} = \{\nu \in Q : a(\nu) = 1\}$. For each positive integer L, let $V_L = L^{-1} \sum_{j=1}^{L} \int_{Q_j^{(L)}} jX_\nu d\nu$ and $W_L = L^{-1} \sum_{j=1}^{L} \int_{S_j^{(L)}} jX_\nu d\nu$ (note that V_L and W_L do not change if $\sum_{j=1}^{L}$ is replaced by $\sum_{j=0}^{L}$). Then by Lemma A.4 and the fact that $a(\nu) - j/L \geq 0$ on $Q_j^{(L)}$, one has that

$$
\begin{aligned}
\left\| \int_Q a(\nu) X_\nu d\nu - V_L \right\|_2 &= \left\| \int_Q a(\nu) X_\nu d\nu - \sum_{j=0}^{L} \int_{Q_j^{(L)}} \frac{j}{L} X_\nu d\nu \right\|_2 \\
&= \left\| \sum_{j=0}^{L} \int_{Q_j^{(L)}} \left(a(\nu) - \frac{j}{L} \right) X_\nu d\nu \right\|_2 \\
&\leq \sum_{j=0}^{L} \left\| \int_{Q_j^{(L)}} \left(a(\nu) - \frac{j}{L} \right) X_\nu d\nu \right\|_2 \\
&\leq \sum_{j=0}^{L} \int_{Q_j^{(L)}} \left(a(\nu) - \frac{j}{L} \right) \|X_\nu\|_2 \, d\nu \\
&\leq \sum_{j=0}^{L} \int_{Q_j^{(L)}} \frac{1}{L} \|X_\nu\|_2 d\nu \\
&= \frac{1}{L} \int_Q \|X_\nu\|_2 \, d\nu.
\end{aligned}
$$

Thus, $\| \int_Q a(\nu) X_\nu d\nu - V_L \|_2 \to 0$ as $L \to \infty$. Analogously, $\| \int_S a(\nu) X_\nu d\nu - W_L \|_2 \to 0$ as $L \to \infty$. Now, by Lemma A.3, it suffices to show that $|\mathrm{E}\, V_L \overline{W_L}| \leq \zeta_c(X,s)\, \lambda(Q \cup S)$. Define $Q^L(j) := Q_j^{(L)} \cup Q_{j+1}^{(L)} \cup \ldots \cup Q_L^{(L)}$ and $S^L(j) := S_j^{(L)} \cup S_{j+1}^{(L)} \cup \ldots \cup S_L^{(L)}$. Then one can write $V_L = L^{-1} \sum_{j=1}^{L} \int_{Q^L(j)} X_\nu d\nu$ and $W_L = L^{-1} \sum_{j=1}^{L} \int_{S^L(j)} X_\nu d\nu$. Thus,

$$
\begin{aligned}
\left| \mathrm{E}\, V_L \overline{W_L} \right| &= \left| \mathrm{E} \left(L^{-2} \sum_{j=1}^{L} \sum_{k=1}^{L} \int_{Q^L(j)} X_\nu d\nu \int_{S^L(k)} \overline{X_\nu} d\nu \right) \right| \\
&\leq L^{-2} \sum_{j=1}^{L} \sum_{k=1}^{L} \left| \mathrm{E} \left(\int_{Q^L(j)} X_\nu d\nu \int_{S^L(k)} \overline{X_\nu} d\nu \right) \right| \\
&\leq L^{-2} \sum_{j=1}^{L} \sum_{k=1}^{L} \zeta_c(X,s)\, \lambda(Q^L(j) \cup S^L(k)) \\
&\leq \zeta_c(X,s)\, \lambda(Q \cup S),
\end{aligned}
$$

and (5.3) holds.

Any complex, Borel function $c(\nu)$ on $Q \cup S$ with $|c(\nu)| \leq 1$ for all $\nu \in Q \cup S$ can be represented by $a_1(\nu) - a_2(\nu) + ia_3(\nu) - ia_4(\nu)$ where $a_j : Q \cup S \to [0,1]$ is a Borel function for each $j \in \{1,2,3,4\}$. Hence, with (5.3) above, one can readily see that (5.4) holds through a simple calculation.

□

Since (5.4) holds for any non-empty, disjoint, bounded Borel subsets Q and S of $\mathbb{R}^d$, and any complex valued Borel function $c(\nu)$ such that $|c(\nu)| \le 1$, (1.17) ensures that

$$\zeta_c(X^{<x>}, s) \le 16\zeta_c(X, s) \text{ for all } s > 0 \text{ and } x \in \mathbb{R}^d . \tag{5.5}$$

Notice also, that whenever $j(\nu)$ is a bounded, complex, Borel function on $\mathbb{R}^d$, then so is $e^{ix\cdot\nu}j(\nu)$. This, along with the definition in (1.16), implies that

$$r'_c(X^{<x>}, s) = r'_c(X, s) \text{ for all } s > 0 \text{ and } x \in \mathbb{R}^d . \tag{5.6}$$

LEMMA 5.2. *Suppose that $\theta := \{\theta_n\}$ and $z := \{z_n\}$ are non-increasing sequences in $[0,1]$ and $[0,\infty]$ respectively such that $\lim_{n\to\infty}\theta_n < 1$ and $\lim_{n\to\infty} z_n = 0$. Then there exists a positive constant $A := A(\theta)$ and a constant $L := L(\varepsilon, \theta, z)$ for each $\varepsilon > 0$ such that the following statement holds. If $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field with $\|X_0\|_2^2 \le 1$, $r'_c(n) \le \theta_n$ and $\zeta_c(n) \le z_n$ for all $n \ge 1$, and the function $T : \mathbb{R}^d \to [0,\infty)$ defined by*

$$T(x) := \mathrm{E}\left|\int_{(0,1)^d} X_\nu^{<x>}\, d\nu\right|^2$$

is integrable, then the following hold:

(a) For all $x \in \mathbb{R}^d$, $f(x) := \lim_{a\to\infty} F(X^{<x>}, a)$ exists in $[0,\infty)$.

(b) For any $\varepsilon > 0$, and any $x \in \mathbb{R}^d$, one has that $|f(x) - F(X^{<x>}, a)| \le \varepsilon$ for every $a \ge L$.

(c) The function f is uniformly continuous on $\mathbb{R}^d$.

(d) The function f is integrable, and in particular, $f(x) \le A \cdot T(x)$ for all $x \in \mathbb{R}^d$.

PROOF. Define $z' := \{z'_n\}$, where $z'_n = 16z_n$ and $\theta' := \{\theta'_n\}$, where $\theta'_1 = 1$ and $\theta'_n = \theta_{n-1}$ for $n \ge 2$. Then $\lim_{n\to\infty} z'_n = 0$ and $\lim_{n\to\infty}\theta'_n < 1$ by assumption. For each $\varepsilon > 0$ define $L := L(\varepsilon, \theta, z')$ as the constant from Lemma 4.4. Define $A := A(\theta', d)$ as the constant from Lemma 2.4. These will be the constants for parts (b) and (d).

Suppose $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field such that $E|X_0|^2 \le 1$, $r'_c(n) \le \theta_n$, and $\zeta_c(n) \le z_n$ for all $n \ge 1$. For any $x \in \mathbb{R}^d$, $q'_c(X^{<x>}, n) \le r'_c(X, n)$ for every integer $n \ge 1$. One can use (5.5) and (5.6), and apply Lemma 4.3 to the random field $X^{<x>}$ and get that $f(x) := \lim_{a\to\infty} F(X^{<x>}, a)$ exists in $[0,\infty)$. Therefore (a) holds. Note that the stronger linear dependence coefficient r'_c is used because the random field $X^{<x>}$ has a complex coefficient.

Fix any $\varepsilon > 0$. Lemma 4.4 implies that for each $x \in \mathbb{R}^d$, $|f(x) - F(X^{<x>}, a)| < \varepsilon$ for all $a \geq L$. Thus, (b) holds with this constant L.

Suppose $\varepsilon > 0$ and let $L := L(\varepsilon/3, \theta, z)$ be the constant obtained from part (b). Then for every $x \in \mathbb{R}^d$ and for every $a \geq L$, $|f(x) - F(X^{<x>}, a)| \leq \varepsilon/3$. The function $F(X^{<x>}, L)$ is uniformly continuous by Lemma A.2 in Appendix A. Let $\delta > 0$ be small enough so that $|F(X^{<x>}, L) - F(X^{<y>}, L)| \leq \varepsilon/3$ if $\|x - y\| < \delta$. A simple application of the triangle inequality now gives

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - F(X^{<x>}, L)| + |F(X^{<x>}, L) - F(X^{<y>}, L)| + |F(X^{<y>}, L) - f(y)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

whenever $\|x - y\| < \delta$. Thus, $f(x)$ is uniformly continuous on $\mathbb{R}^d$ and (c) holds.

Suppose $x \in \mathbb{R}^d$, and let $Y^{<x>}$ be the CCWS random field defined in (5.2). Notice that $r'_D(Y^{<x>}, n) \leq r'_c(X^{<x>}, n-1) \leq \theta_{n-1} = \theta'_n$ for all $n \geq 2$. With the comment following (5.2) and the constant A above, Lemma 2.4 gives

$$\begin{aligned} F(X^{<x>}, n) &= n^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X^{<x>}, n)|^2 \\ &= n^{-d}\,\mathrm{E}\,|S(Y^{<x>}, n)|^2 \\ &\leq A \cdot \|Y_0^{<x>}\|_2^2 \\ &= A \cdot \|Y_1^{<x>}\|_2^2 \\ &= A \cdot T(x). \end{aligned}$$

If one takes a limit as $n \to \infty$, one has $f(x) \leq A \cdot T(x)$ by part (a). Since $T(x)$ is integrable and $x \in \mathbb{R}^d$ is arbitrary, part (d) holds for this constant A and the proof is complete. □

Lemma 5.2 obtains a function f, but does not show that it is the spectral density for the random field X. Using part (a) of Lemma 5.2 and (A.1) in Appendix A, one can get

$$
\begin{aligned}
f(x) &= \lim_{L\to\infty} F(X^{<x>}, L) \\
&= \lim_{L\to\infty} L^{-d} E \left| \int_{(0,L)^d} e^{-ix\cdot\nu} X_\nu d\nu \right|^2 \\
&= \lim_{L\to\infty} L^{-d} \int_{[-L,L]^d} e^{-ix\cdot\nu} \left(\prod_{j=1}^{d} (L - |\nu_j|) \right) \gamma(\nu) d\nu \\
&= \lim_{L\to\infty} \int_{[-L,L]^d} e^{-ix\cdot\nu} \left(\prod_{j=1}^{d} \left(1 - \frac{|\nu_j|}{L}\right) \right) \gamma(\nu) d\nu \\
&= \lim_{L\to\infty} \int\limits_{\mathbb{R}^d} e^{-ix\cdot\nu} \mathbf{1}_{[-L,L]^d}(\nu) \cdot \prod_{j=1}^{d} \left(1 - \frac{|\nu_j|}{L}\right) \gamma(\nu) d\nu. \qquad (5.7)
\end{aligned}
$$

The integrand in (5.7) is dominated by $|\gamma(\nu)|$. Since γ is not known to be integrable, one cannot use the inversion theorem to show that f is in fact the spectral density for the random field X (look at Definition 1.3 and replace g by f). The following chapter will create a CCWS random field $X^{(\rho)}$ for $\rho \in (0,1)$ with the property that $\gamma(X^{(\rho)}, \nu) = \gamma(X, \nu) \cdot \rho^{\sum |\nu_i|}$. This new random field will be CCWS and satisfy Lemma 5.2. Replacing X with $X^{(\rho)}$ (f_ρ instead of f) and inserting $\rho^{\sum |\nu_i|}$ in (5.7), one can get a dominating function of $\gamma(0) \cdot \rho^{\sum |\nu_i|}$, which is integrable. Then, the inversion theorem can be used to show that f_ρ is the spectral density of $X^{(\rho)}$ for each $\rho \in (0,1)$. Taking $\rho \to 1^-$ and using Lebesgue's Dominated Convergence Theorem will help show that f is the spectral density function of the original random field X.

CHAPTER 6

The Random Field $X^{(\rho)}$

For a given $\rho \in (0,1)$, the random field $X^{(\rho)} := (X^{(\rho)}_\nu : \nu \in \mathbb{R}^d)$ will make use of standard independent Poisson processes with parameter $\lambda := -\ln\rho$ (mean $1/\lambda$).

Fix a $\rho \in (0,1)$. Let $(\Omega^{(\rho)}, \mathcal{F}^{(\rho)}, P^{(\rho)})$ be a large enough probability space (use Theorem 20.4 in [**1**]) so that for each $n \in \mathbb{N}$ and $j \in \{1,2,\ldots,d\}$, one can have families $\tau_{n,j}$ and $\tau'_{n,j}$ of random variables defined on $(\Omega^{(\rho)}, \mathcal{F}^{(\rho)}, P^{(\rho)})$ such that all of the random variables in the entire collection are independent of each other and follow an exponential distribution with parameter $-\ln\rho$ (mean $-1/\ln\rho$).

For each $j \in \{1,2,\ldots,d\}$, define the random sequence $(\ldots, S^j_{-1}, S^j_0, S^j_1, \ldots)$ on $(\Omega^{(\rho)}, \mathcal{F}^{(\rho)}, P^{(\rho)})$ by $S^j_n(\omega') = \sum_{k=1}^n \tau_{k,j}(\omega')$ if $n > 0$ and $S^j_n(\omega') = \sum_{k=1}^{-n+1} -\tau'_{k,j}(\omega')$ if $n \le 0$. Then for all $a \in \mathbb{R}$ and each $j \in \{1,2,\ldots,d\}$, let

$$N^j_a(\omega') = \max[n \in \mathbb{Z} : S^j_n(\omega') \le a] \tag{6.1}$$

([**1**][pg. 298]). For each $j \in \{1,2,\ldots,d\}$, $(N^j_a : a \in \mathbb{R})$ is a Poisson process with rate $-\ln(\rho)$ (defined on the probability space $(\Omega^{(\rho)}, \mathcal{F}^{(\rho)}, P^{(\rho)})$). One can assume without loss of generality that for every $\omega' \in \Omega^{(\rho)}$ and every $j \in \{1,2,\ldots,d\}$, $N^j_a(\omega') \to -\infty$ as $a \to -\infty$ and $N^j_a(\omega') \to \infty$ as $a \to \infty$. Now, when $r := (r(1), r(2), \ldots, r(d)) \in \mathbb{R}^d$ and $j \in \{1,2,\ldots,d\}$, let $N_r(\omega') := (N^1_{r(1)}(\omega'), N^2_{r(2)}(\omega'), \ldots, N^d_{r(d)}(\omega'))$ where each $N^j_{r(j)}(\omega')$ is defined as in (6.1).

Enlarging the probability space $(\Omega, \mathcal{F}, P)$ if necessary, for each $n \in \mathbb{Z}^d$, one can define the random field W^n by

$$W^n := (W^n_r : r \in \mathbb{R}^d),$$

so that X and all the W^n are independent and identically distributed (see section 5 of Appendix A). Now, for each fixed $\rho \in (0,1)$, define the random field $X^{(\rho)} := (X^{(\rho)}_r : r \in \mathbb{R}^d)$ on the product space $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P}) := (\Omega \times \Omega^{(\rho)}, \mathcal{F} \times \mathcal{F}^{(\rho)}, P \times P^{(\rho)})$ by

$$X^{(\rho)}_r(\omega, \omega') = W^{N_r(\omega')}_r(\omega).$$

In other words, the random field $X^{(\rho)}$ is defined on d-dimensional blocks where each vertex is a d-tuple of points in each of the d Poisson processes. Every block then contains a new, independent copy of X, namely W^{N_r}.

Since there are three probability spaces present, the notation E_P, $\mathrm{E}_{P^{(\rho)}} := \mathrm{E}_\rho$, and $\mathrm{E}_{\mathbf{P}}$ will be used to distinguish between taking expected values with respect to the probability spaces $(\Omega, \mathcal{F}, P)$, $(\Omega^{(\rho)}, \mathcal{F}^{(\rho)}, P^{(\rho)})$, and $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ respectively.

Let $\mathbf{1}(\cdot)$ denote the indicator function and notice that

$$\begin{aligned}
\mathrm{E}_{\mathbf{P}}\,|X_r^{(\rho)}|^2 &= \mathrm{E}_{\mathbf{P}}\,|W_r^{N_r}|^2 \\
&= \sum_{j\in\mathbb{Z}^d} \mathrm{E}_{\mathbf{P}}\left[|W_r^{N_r}|^2 \mathbf{1}(N_r = j)\right] \\
&= \sum_{j\in\mathbb{Z}^d} \mathrm{E}_{\mathbf{P}}\left[|W_r^{j}|^2 \mathbf{1}(N_r = j)\right] \\
&= \sum_{j\in\mathbb{Z}^d} \mathrm{E}_P\,|W_r^{j}|^2 \cdot E_\rho(\mathbf{1}(N_r = j)) \\
&= \sum_{j\in\mathbb{Z}^d} \mathrm{E}_P\,|X_r|^2 \cdot P^{(\rho)}(N_r = j) \\
&= \mathrm{E}_P\,|X_0|^2 \cdot \sum_{j\in\mathbb{Z}^d} P^{(\rho)}(N_r = j) \\
&= \mathrm{E}_P\,|X_0|^2 \cdot 1 \\
&< \infty.
\end{aligned}$$

Hence, the random field $X^{(\rho)}$ has finite second moments. Using the fact that the original random field X is centered, one can show that the random field $X^{(\rho)}$ is also centered as follows:

$$\begin{aligned}
\mathrm{E}_{\mathbf{P}}\,X_r^{(\rho)} &= \mathrm{E}_{\mathbf{P}}\,W_r^{N_r} \\
&= \sum_{j\in\mathbb{Z}^d} \mathrm{E}_{\mathbf{P}}\left[W_r^{N_r} \cdot \mathbf{1}(N_r = j)\right] \\
&= \sum_{j\in\mathbb{Z}^d} \mathrm{E}_{\mathbf{P}}\left[W_r^{j} \cdot \mathbf{1}(N_r = j)\right] \\
&= \sum_{j\in\mathbb{Z}^d} \mathrm{E}_P\,W_r^{j} \cdot \mathrm{E}_\rho(\mathbf{1}(N_r = j)) \\
&= \sum_{j\in\mathbb{Z}^d} \mathrm{E}_P\,X_r \cdot P^{(\rho)}(N_r = j) \\
&= 0.
\end{aligned}$$

For $\nu \in \mathbb{R}^d$, let $\nu_\bullet := \sum_{j=1}^d \nu_j$. Also, let $|\nu|_\bullet := \sum_{j=1}^d |\nu_j|$. To see that the random field $X^{(\rho)}$ is weakly stationary, first observe that for $r, s \in \mathbb{R}^d$ and distinct $j, k \in \mathbb{Z}^d$, $\mathrm{E}_P\,W_r^j \overline{W_s^k} =$

$\mathrm{E}_P\, W_r^j \cdot \mathrm{E}_P\, \overline{W_s^k} = 0 \cdot 0 = 0$. Let $r, s \in \mathbb{R}^d$ be arbitrary and note that

$$\begin{aligned}
\mathrm{E}_{\mathbf{P}}\, X_r^{(\rho)} \overline{X_s^{(\rho)}} &= \mathrm{E}_{\mathbf{P}}\, W_r^{N_r} \overline{W_s^{N_s}} \\
&= \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \mathrm{E}_{\mathbf{P}} \left(W_r^{N_r} \mathbf{1}(N_r = j) \overline{W_s^{N_s} \mathbf{1}(N_s = k)} \right) \\
&= \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \mathrm{E}_{\mathbf{P}} \left(W_r^{j} \mathbf{1}(N_r = j) \overline{W_s^{k}} \mathbf{1}(N_s = k) \right) \\
&= \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \mathrm{E}_P\, W_r^j \overline{W_s^k} \cdot \mathrm{E}_\rho \left(\mathbf{1}(N_r = j) \mathbf{1}(N_s = k) \right) \\
&= \sum_{j \in \mathbb{Z}^d} \mathrm{E}_P\, W_r^j \overline{W_s^j}\, \mathrm{E}_\rho (\mathbf{1}(N_r = j \text{ and } N_s = j)) \\
&= \sum_{j \in \mathbb{Z}^d} \mathrm{E}_P\, X_r \overline{X_s} \cdot P^{(\rho)}(N_r = N_s = j) \\
&= \gamma(X, r-s) \cdot P^{(\rho)}(N_r = N_s) \\
&= \gamma(X, r-s) \cdot P^{(\rho)}(N_{r(1)} = N_{s(1)}, N_{r(2)} = N_{s(2)}, \ldots, N_{r(d)} = N_{s(d)}) \\
&= \gamma(X, r-s) \cdot P^{(\rho)}(N_{r(1)} = N_{s(1)}) \cdot \ldots \cdot P^{(\rho)}(N_{r(d)} = N_{s(d)}) \\
&= \gamma(X, r-s) \cdot P^{(\rho)}(N_{r(1)} - N_{s(1)} = 0) \cdot \ldots \cdot P^{(\rho)}(N_{r(d)} - N_{s(d)} = 0) \\
&= \gamma(X, r-s) \prod_{j=1}^{d} \rho^{|r(j) - s(j)|} \\
&= \gamma(X, r-s) \cdot \rho^{|r-s|_\bullet},
\end{aligned}$$

where the second-to-last equality is done in Billingsley [**1**] (23.9). Since the covariance function depends only on the difference of the subscripts, we have that $X^{(\rho)}$ is weakly stationary. Hence, $X^{(\rho)}$ is a CCWS random field and one can write $\gamma(X^{(\rho)}, \nu) = \gamma(X, \nu) \cdot \rho^{|\nu|_\bullet}$.

It will be shown that the CCWS random field $X^{(\rho)}$ satisfies Lemma 5.2 (assuming X does). Multiplying by a constant, one can assume without loss of generality that $\|X_0^{(\rho)}\|_2^2 = \|X_0\|_2^2 \le 1$. Now, it will be shown that $\zeta_c(X^{(\rho)}, s) \to 0$ as $s \to \infty$, which will follow from $\zeta_c(X^{(\rho)}, s) \le \zeta_c(X, s)$. From the construction of $X^{(\rho)}$, this is intuitively obvious, since $X^{(\rho)}$ is less dependent than X (all W^n's are independent), and ζ is a linear dependence coefficient. Analogously, $r'_c(X^{(\rho)}, s) \le r'_c(X, s)$. However, these inequalities are tedious to show, which is what follows. One may want to skip these details and go to the end of this chapter where it will be shown that $T^{(\rho)}$ is integrable.

For each $n \in \mathbb{Z}^d$ and $\omega' \in \Omega^{(\rho)}$, define the d-dimensional block $I_n(\omega') := \prod_{j=1}^d [S^j_{n(j)}(\omega'), S^j_{n(j)+1}(\omega'))$. Suppose $s > 0$, and let Q and S be bounded Borel sets in $\mathbb{R}^d$ that satisfy (1.18). Define the sets $Q^*(\omega') := \{n \in \mathbb{Z}^d : \lambda(I_n(\omega') \cap Q) > 0\}$ and $S^*(\omega') := \{n \in \mathbb{Z}^d :$

$\lambda(I_n(\omega') \cap S) > 0\}$, and notice that both of these must be finite. To help simplify things, define $\boldsymbol{\omega} := (\omega, \omega') \in \mathbf{\Omega}$. Then, using Fubini,

$$\begin{aligned}
&\left| \mathrm{E}_{\mathbf{P}} \left(\int_Q X_\nu^{(\rho)} d\nu \right) \left(\int_S \overline{X_r^{(\rho)}} dr \right) \right| \\
&= \left| \int_{\mathbf{\Omega}} \left(\int_Q X_\nu^{(\rho)}(\boldsymbol{\omega}) d\nu \int_S \overline{X_r^{(\rho)}(\boldsymbol{\omega})} dr \right) d\mathbf{P}(\boldsymbol{\omega}) \right| \\
&= \left| \int_{\Omega^{(\rho)}} \int_\Omega \left(\int_Q X_\nu^{(\rho)}(\boldsymbol{\omega}) d\nu \int_S \overline{X_r^{(\rho)}(\boldsymbol{\omega})} dr \right) dP(\omega) dP^{(\rho)}(\omega') \right| \\
&\leq \int_{\Omega^{(\rho)}} \left| \int_\Omega \left(\int_Q X_\nu^{(\rho)}(\boldsymbol{\omega}) d\nu \int_S \overline{X_r^{(\rho)}(\boldsymbol{\omega})} dr \right) dP(\omega) \right| dP^{(\rho)}(\omega') \qquad (6.3)
\end{aligned}$$

To save space, ω' will be fixed and the inside integral of (6.3) (without the modulus) will be simplified. In the calculations below, remember that $X_r^{(\rho)}(\boldsymbol{\omega}) = W_r^n(\omega)$ for r inside the block $I_n(\omega')$, and that all the W^n's are independent and have the same distribution as X. Since the ω' is fixed, let $Q^* = Q^*(\omega')$, $S^* = S^*(\omega')$, and $I_n = I_n(\omega')$, so that

$$\begin{aligned}
&\int_\Omega \left(\int_Q X_\nu^{(\rho)}(\omega, \omega') d\nu \int_S \overline{X_r^{(\rho)}(\omega, \omega')} dr \right) dP(\omega) \\
&= \int_\Omega \left(\sum_{n \in Q^*} \sum_{m \in S^*} \int_{Q \cap I_n} X_\nu^{(\rho)}(\omega, \omega') d\nu \int_{S \cap I_m} \overline{X_r^{(\rho)}(\omega, \omega')} dr \right) dP(\omega) \\
&= \int_\Omega \left(\sum_{n \in Q^*} \sum_{m \in S^*} \int_{Q \cap I_n} W_\nu^n(\omega) d\nu \int_{S \cap I_m} \overline{W_r^m(\omega)} dr \right) dP(\omega) \\
&= \sum_{n \in Q^*} \sum_{m \in S^*} \int_\Omega \left(\int_{Q \cap I_n} W_\nu^n(\omega) d\nu \int_{S \cap I_m} \overline{W_r^m(\omega)} dr \right) dP(\omega) \\
&= \sum_{n \in Q^* \cap S^*} \int_\Omega \left(\int_{Q \cap I_n} W_\nu^n(\omega) d\nu \int_{S \cap I_n} \overline{W_r^n(\omega)} dr \right) dP(\omega) + 0 \\
&= \sum_{n \in Q^* \cap S^*} \int_\Omega \left(\int_{Q \cap I_n} X_\nu(\omega) d\nu \int_{S \cap I_n} \overline{X_r(\omega)} dr \right) dP(\omega) \\
&= \sum_{n \in Q^* \cap S^*} \mathrm{E}_P \left(\int_{Q \cap I_n} X_\nu(\omega) d\nu \int_{S \cap I_n} \overline{X_r(\omega)} dr \right).
\end{aligned}$$

Look at the modulus now, and get

$$\begin{aligned}
&\left| \sum_{n \in Q^* \cap S^*} \mathrm{E}_P \left(\int_{Q \cap I_n} X_\nu(\omega) d\nu \int_{S \cap I_n} \overline{X_r(\omega)} dr \right) \right| \\
&\leq \sum_{n \in Q^* \cap S^*} \left| \mathrm{E}_P \left(\int_{Q \cap I_n} X_\nu(\omega) d\nu \int_{S \cap I_n} \overline{X_r(\omega)} dr \right) \right| \\
&\leq \sum_{n \in Q^* \cap S^*} \zeta_c(X, s)\, \lambda\left((Q \cup S) \cap I_n\right) \\
&\leq \zeta_c(X, s)\, \lambda(Q \cup S).
\end{aligned}$$

This is a constant, so substituting back into (6.3), one gets that

$$\begin{aligned}\left|\mathrm{E}_{\mathbf{P}}\left(\int_Q X_\nu^{(\rho)} d\nu\right)\left(\int_S \overline{X_r^{(\rho)}} dr\right)\right| &\leq \int_{\Omega^{(\rho)}} \zeta_c(X,s)\,\lambda(Q\cup S) dP^{(\rho)}(\omega') \\ &= \zeta_c(X,s)\,\lambda(Q\cup S).\end{aligned}$$

Since this is true for all bounded Borel sets Q and S of $\mathbb{R}^d$ such that (1.18) holds, it follows that $\zeta_c(X^{(\rho)},s) \leq \zeta_c(X,s)$.

Next, one needs to show that $r'_c(X^{(\rho)},s) < 1$ for some $s > 0$. Choose an s such that $r'_c(X,s) < 1$, and this s will suffice, since it will be shown that $r'_c(X^{(\rho)},s) \leq r'_c(X,s)$. This will take a similar calculation. Take Q and S to be any bounded Borel sets satisfying (1.18), and use the same Q^*, S^*, and I_n defined before. Also, let $j(\nu)$ be an arbitrary Borel, bounded, complex function on $\mathbb{R}^d$. Then,

$$\begin{aligned}&\left|\mathrm{E}_{\mathbf{P}}\left(\int_Q j(\nu)X_\nu^{(\rho)} d\nu \int_S \overline{j(r)X_r^{(\rho)}} dr\right)\right| \\ &= \left|\int_{\boldsymbol{\Omega}}\left(\int_Q j(\nu)X_\nu^{(\rho)}(\boldsymbol{\omega}) d\nu \int_S \overline{j(r)X_r^{(\rho)}(\boldsymbol{\omega})} dr\right) d\mathbf{P}(\boldsymbol{\omega})\right| \\ &= \left|\int_{\Omega^{(\rho)}}\int_\Omega\left(\int_Q j(\nu)X_\nu^{(\rho)}(\boldsymbol{\omega}) d\nu \int_S \overline{j(r)X_r^{(\rho)}(\boldsymbol{\omega})} dr\right) dP(\omega) dP^{(\rho)}(\omega')\right| \\ &\leq \int_{\Omega^{(\rho)}}\left|\int_\Omega\left(\int_Q j(\nu)X_\nu^{(\rho)}(\boldsymbol{\omega}) d\nu \int_S \overline{j(r)X_r^{(\rho)}(\boldsymbol{\omega})} dr\right) dP(\omega)\right| dP^{(\rho)}(\omega') \qquad (6.4)\end{aligned}$$

Again, for a given $\omega' \in \Omega^{(\rho)}$, simplify the inside integral of (6.4) (without the modulus) by using the same ideas as in the previous calculation,

$$\begin{aligned}&\int_\Omega\left(\int_Q j(\nu)X_\nu^{(\rho)}(\omega,\omega') d\nu \int_S \overline{j(r)X_r^{(\rho)}(\omega,\omega')} dr\right) dP(\omega) \\ &= \int_\Omega\left(\sum_{n\in Q^*}\int_{Q\cap I_n} j(\nu)X_\nu^{(\rho)}(\omega,\omega') d\nu \sum_{m\in S^*}\int_{S\cap I_m} \overline{j(r)X_r^{(\rho)}(\omega,\omega')} dr\right) dP(\omega) \\ &= \sum_{n\in Q^*}\sum_{m\in S^*}\int_\Omega\left(\int_{Q\cap I_n} j(\nu)X_\nu^{(\rho)}(\omega,\omega') d\nu \int_{S\cap I_m} \overline{j(r)X_r^{(\rho)}(\omega,\omega')} dr\right) dP(\omega) \\ &= \sum_{n\in Q^*}\sum_{m\in S^*}\int_\Omega\left(\int_{Q\cap I_n} j(\nu)W_\nu^n(\omega) d\nu \int_{S\cap I_m} \overline{j(r)W_r^m(\omega)} dr\right) dP(\omega) \\ &= \sum_{n\in Q^*\cap S^*}\int_\Omega\left(\int_{Q\cap I_n} j(\nu)W_\nu^n(\omega) d\nu \int_{S\cap I_n} \overline{j(r)W_r^n(\omega)} dr\right) dP(\omega) + 0 \\ &= \sum_{n\in Q^*\cap S^*}\int_\Omega\left(\int_{Q\cap I_n} j(\nu)X_\nu(\omega) d\nu \int_{S\cap I_n} \overline{j(r)X_r(\omega)} dr\right) dP(\omega) \\ &= \sum_{n\in Q^*\cap S^*}\mathrm{E}_P\left(\int_{Q\cap I_n} j(\nu)X_\nu(\omega) d\nu \int_{S\cap I_n} \overline{j(r)X_r(\omega)} dr\right).\end{aligned}$$

Put the modulus back in now, and get

$$\begin{aligned}
&\left|\sum_{n\in Q^*\cap S^*} \mathrm{E}_P\left(\int_{Q\cap I_n} j(\nu)X_\nu(\omega)d\nu \int_{S\cap I_n} \overline{j(r)X_r(\omega)}dr\right)\right| \\
&\quad\leq \sum_{n\in Q^*\cap S^*} \left|\mathrm{E}_P\left(\int_{Q\cap I_n} j(\nu)X_\nu(\omega)d\nu \int_{S\cap I_n} \overline{j(r)X_r(\omega)}dr\right)\right| \\
&\quad\leq \sum_{n\in Q^*\cap S^*} r'_c(X,s)\left\|\int_{Q\cap I_n} j(\nu)W^n_\nu d\nu\right\|_2 \left\|\int_{S\cap I_n} j(r)W^n_r dr\right\|_2 .
\end{aligned}$$

Substitute this into (6.4). Since $r'_c(X,s)$ is a constant, using Cauchy's inequality on (6.4) will give

$$\begin{aligned}
&\left|\mathrm{E}_{\mathbf{P}}\left(\int_Q j(\nu)X^{(\rho)}_\nu d\nu \int_S \overline{j(r)X^{(\rho)}_r}dr\right)\right| \\
&\quad\leq r'_c(X,s)\int_{\Omega^{(\rho)}}\left(\sum_{n\in Q^*\cap S^*}\left\|\int_{Q\cap I_n} j(\nu)W^n_\nu d\nu\right\|_2\left\|\int_{S\cap I_n} j(r)W^n_r dr\right\|_2\right)dP^{(\rho)}(\omega') \\
&\quad\leq r'_c(X,s)\int_{\Omega^{(\rho)}}\prod_{\Lambda\in\{Q,S\}}\left(\sum_{n\in Q^*\cap S^*}\left\|\int_{\Lambda\cap I_n} j(\nu)W^n_\nu d\nu\right\|_2^2\right)^{1/2} dP^{(\rho)}(\omega') \\
&\quad\leq r'_c(X,s)\prod_{\Lambda\in\{Q,S\}}\left(\int_{\Omega^{(\rho)}}\sum_{n\in Q^*\cap S^*}\left\|\int_{\Lambda\cap I_n} j(\nu)W^n_\nu d\nu\right\|_2^2 dP^{(\rho)}(\omega')\right)^{1/2} \\
&\quad\leq r'_c(X,s)\prod_{\Lambda\in\{Q,S\}}\left(\int_{\Omega^{(\rho)}}\sum_{n\in \Lambda^*}\left\|\int_{\Lambda\cap I_n} j(\nu)W^n_\nu d\nu\right\|_2^2 dP^{(\rho)}(\omega')\right)^{1/2} . \qquad (6.5)
\end{aligned}$$

If $\| \int_\Lambda j(\nu)X^{(\rho)}_\nu d\nu\|_2^2$ is equal to the expression inside the large set of parenthesis in (6.5), then (1.16) would imply $r'_c(X^{(\rho)},s) \leq r'_c(X,s) < 1$ since Q and S were arbitrary Borel sets satisfying

(1.18). This equality is given by

$$\begin{aligned}
&\left\| \int_{\Lambda} j(\nu) X_\nu^{(\rho)} d\nu \right\|_2^2 \\
&= \int_{\mathbf{\Omega}} \left| \int_{\Lambda} j(\nu) X_\nu^{(\rho)} d\nu \right|^2 d\mathbf{P}(\boldsymbol{\omega}) \\
&= \int_{\mathbf{\Omega}} \left| \sum_{n \in \Lambda^*} \int_{\Lambda \cap I_n} j(\nu) X_\nu^{(\rho)} d\nu \right|^2 d\mathbf{P}(\boldsymbol{\omega}) \\
&= \int_{\mathbf{\Omega}} \left(\sum_{n \in \Lambda^*} \int_{\Lambda \cap I_n} j(\nu) X_\nu^{(\rho)}(\boldsymbol{\omega}) d\nu \right) \left(\sum_{m \in \Lambda^*} \int_{\Lambda \cap I_m} \overline{j(r) X_r^{(\rho)}(\boldsymbol{\omega})} dr \right) d\mathbf{P}(\boldsymbol{\omega}) \\
&= \int_{\mathbf{\Omega}} \sum_{n,m \in \Lambda^*} \left(\int_{\Lambda \cap I_n} j(\nu) X_\nu^{(\rho)}(\boldsymbol{\omega}) d\nu \right) \left(\int_{\Lambda \cap I_m} \overline{j(r) X_r^{(\rho)}(\boldsymbol{\omega})} dr \right) d\mathbf{P}(\boldsymbol{\omega}) \\
&= \int_{\Omega^{(\rho)}} \sum_{n,m \in \Lambda^*} \int_{\Omega} \left(\int_{\Lambda \cap I_n} j(\nu) X_\nu^{(\rho)}(\boldsymbol{\omega}) d\nu \int_{\Lambda \cap I_m} \overline{j(r) X_r^{(\rho)}(\boldsymbol{\omega})} dr \right) dP(\omega) dP^{(\rho)}(\omega') \\
&= \int_{\Omega^{(\rho)}} \sum_{n,m \in \Lambda^*} \int_{\Omega} \left(\int_{\Lambda \cap I_n} j(\nu) W_\nu^n(\omega) d\nu \int_{\Lambda \cap I_m} \overline{j(r) W_r^m(\omega)} dr \right) dP(\omega) dP^{(\rho)}(\omega') \\
&= \int_{\Omega^{(\rho)}} \sum_{n \in \Lambda^*} \int_{\Omega} \left| \int_{\Lambda \cap I_n} j(\nu) W_\nu^n(\omega) d\nu \right|^2 dP(\omega) dP^{(\rho)}(\omega') \\
&= \int_{\Omega^{(\rho)}} \sum_{n \in \Lambda^*} \left\| \int_{\Lambda \cap I_n} j(\nu) W_\nu^n d\nu \right\|_2^2 dP^{(\rho)}(\omega').
\end{aligned}$$

Thus, one can now see that $r'_c(X^{(\rho)}, s) \leq r_c(X, s)$.

Recall that $\mathrm{E}_{\mathbf{P}}\, X_\nu^{(\rho)} \overline{X_0^{(\rho)}} = \gamma(X^{(\rho)}, \nu) = \rho^{|\nu|_\bullet} \gamma(X, \nu)$. For a fixed $\rho \in (0,1)$, let

$$T^{(\rho)}(x) = \int_{[-1,1]^d} e^{-ix\cdot\nu} \left(\prod_{i=1}^{d} (1 - |\nu_i|) \right) \gamma(X^{(\rho)}, \nu) d\nu$$

for $x \in \mathbb{R}^d$. The last thing that needs to be shown for the random field $X^{(\rho)}$ to satisfy Lemma 5.2 (assuming X does) is that $T^{(\rho)}$ is integrable.

Let $g(\nu) = \mathbf{1}_{[-1,1]^d}(\nu) \left(\prod_{j=1}^{d} (1 - |\nu_j|) \right) \gamma(X, \nu)$. Let $\mu_d(\cdot)$ denote a rescaled Lebesgue measure on $\mathbb{R}^d$ defined by $d\mu_d(x) = (2\pi)^{-d/2} dx$ (as in Appendix B). Notice that from (4.1), $T(x) = \int_{\mathbb{R}^d} e^{-ix\cdot\nu} g(\nu) d\nu$. The function g is bounded with bounded support and therefore is integrable. It is assumed that $T(x)$ is integrable. Define $\mathcal{T}(x) := (2\pi)^{-d/2} T(x)$ and $\mathcal{T}^{(\rho)}(x) := (2\pi)^{-d/2} T^{(\rho)}(x)$. Since $T(x)$ is integrable, $\mathcal{T}(x)$ is integrable. Using an inversion theorem from Fourier analysis, one has that

$$g(x) = \int_{\mathbb{R}^d} e^{ix\cdot\nu} \mathcal{T}(\nu) d\mu_d(\nu)$$

for almost every $x \in \mathbb{R}^d$. With the definitions in Section 1 of Appendix B (slightly different from standard definitions), one has $g(x) = \widehat{\mathcal{T}}(x)$ for almost every $x \in \mathbb{R}^d$. In fact, $g = \widehat{\mathcal{T}}(x)$ for all $x \in \mathbb{R}^d$ since they are continuous and equal a.e.

Refer to Section 2 of Appendix B for the following definitions.

Let $\lambda = -\ln(\rho)$ for $\rho \in (0,1)$. Then define $H_\rho(\nu) = H(\lambda\,\nu) = \rho^{|\nu|_\bullet}$, and let

$$h_\rho(x) = \int_{\mathbb{R}^d} H_\rho(\nu) e^{-ix\cdot\nu} d\mu_d(\nu).$$

Refer to Section 2 of Appendix B to see that both H_ρ and h_ρ are continuous, integrable, even functions. By the inversion theorem, one has

$$H_\rho(x) = \int_{\mathbb{R}^d} h_\rho(\nu) e^{ix\cdot\nu} d\mu_d(\nu)$$

for every $x \in \mathbb{R}^d$. Thus, $H_\rho(x) = \hat{h}_\rho(x)$ for every $x \in \mathbb{R}^d$.

Since $\mathcal{T}$ is integrable and h_ρ is integrable for all $\rho \in (0,1)$, one has that $\mathcal{T} * h_\rho$ is integrable for all $\rho \in (0,1)$. Now, using Theorem B.1 and the calculations above, one can see that $\widehat{\mathcal{T} * h_\rho} = \widehat{\mathcal{T}} \cdot \hat{h}_\rho = g \cdot H_\rho$. Since $g \cdot H_\rho$ is integrable, the inversion theorem gives

$$\begin{aligned}
(\mathcal{T} * h_\rho)(x) &= \int\limits_{\mathbb{R}^d} e^{-ix\cdot\nu} g(\nu) H_\rho(\nu) d\mu_d(\nu) \\
&= \int\limits_{[-1,1]^d} e^{-ix\cdot\nu} \left(\prod_{j=1}^{d} (1 - |\nu_j|) \right) \gamma(X,\nu) \rho^{|\nu|_\bullet} d\mu_d(\nu) \\
&= \mathcal{T}^{(\rho)}(x)
\end{aligned}$$

for every $x \in \mathbb{R}^d$. Hence, $\mathcal{T}^{(\rho)}$ is integrable for $\rho \in (0,1)$ which implies that $T^{(\rho)}$ is also. Thus, $X^{(\rho)}$ satisfies Lemma 5.2 for all $\rho \in (0,1)$.

It will be useful to note that Theorem B.3 in Appendix B implies that $\|\mathcal{T} * h_\rho - \mathcal{T}\|_1 \to 0$ as $\rho \to 1^-$, which is the same as $\|\mathcal{T}^{(\rho)} - \mathcal{T}\|_1 \to 0$ as $\rho \to 1^-$. In particular, one has that

$$\|T^{(\rho)} - T\|_1 \to 0 \text{ as } \rho \to 1^-. \tag{6.6}$$

CHAPTER 7

Proof of Theorem 4.1

Let $X := (X_\nu : \nu \in \mathbb{R}^d)$ be a non-degenerate, CCWS random field such that $\zeta_c(s) \to 0$ as $s \to \infty$, and $r_c'(a) < 1$ for some $a > 0$. Also, suppose that $T(x)$ (as defined in Theorem 4.1 and in (4.1)) is integrable. Without loss of generality, assume that $\|X_0\|_2^2 \leq 1$ (multiply the field by appropriate constant if needed). Define the non-increasing sequences $\theta := \{\theta_n\}$ and $z := \{z_n\}$ by $\theta_n := r_c'(X, n)$ and $z_n := \zeta_c(X, n)$. The results of Chapter 6 show that the CCWS random field $X^{(\rho)}$ (defined in Chapter 6) satisfies Lemma 5.2 under these two sequences for each $\rho \in (0, 1)$.

The proof is trivial in the degenerate case, so assume that $0 < \|X_0\|_2 \leq 1$. For each $x \in \mathbb{R}^d$, let $X^{(\rho,x)} := (X_\nu^{(\rho,x)} : \nu \in \mathbb{R}^d)$ where $X_\nu^{(\rho,x)} = e^{-ix\cdot\nu} X_\nu^{(\rho)}$. Then Lemma 5.2 implies that for every $x \in \mathbb{R}^d$, both $f_\rho(x) := \lim_{a\to\infty} F(X^{(\rho,x)}, a)$ and $f(x) := \lim_{a\to\infty} F(X^{<x>}, a)$ exist. Lemma 5.2 also implies that the functions f_ρ and f are continuous and integrable. It will now be shown that $f_\rho(x) \to f(x)$ uniformly as $\rho \to 1^-$. It will suffice to show that for each $\varepsilon > 0$, there exists a $\rho_1 \in (0, 1)$ such that $|f_\rho(x) - f(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^d$ whenever $\rho \in [\rho_1, 1)$.

Fix any $\varepsilon > 0$, and let $L := L(\varepsilon/3, \theta, z)$ be the constant from Lemma 5.2. Then $|f_\rho(x) - F(X^{(\rho,x)}, a)| \leq \varepsilon/3$ and $|f(x) - F(X^{<x>}, a)| \leq \varepsilon/3$ for every $a \geq L$. Let $\rho_1 \in (0, 1)$ be such that $|1 - \rho_1^{dL}| \leq \varepsilon/(3(2L)^d \|X_0\|_2^2)$. Now, refer to (A.1), and note that

$$
\begin{aligned}
&\left|F(X^{(\rho,x)},L) - F(X^{<x>},L)\right| \\
&= \left|L^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X^{(\rho,x)},L)|^2 - L^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X^{<x>},L)|^2\right| \\
&= \left|L^{-d}\int\limits_{[-L,L]^d} e^{-ix\cdot r}\prod_{i=1}^{d}(L-|r_i|)\gamma(r)\rho^{|r|\bullet}dr - L^{-d}\int\limits_{[-L,L]^d} e^{-ix\cdot r}\prod_{i=1}^{d}(L-|r_i|)\gamma(r)dr\right| \\
&= \left|\int\limits_{[-L,L]^d} e^{-ix\cdot r}\prod_{i=1}^{d}\left(1-\frac{|r_i|}{L}\right)\gamma(r)\rho^{|r|\bullet}dr - \int\limits_{[-L,L]^d} e^{-ix\cdot r}\prod_{i=1}^{d}\left(1-\frac{|r_i|}{L}\right)\gamma(r)dr\right| \\
&= \left|\int\limits_{[-L,L]^d} e^{-ix\cdot r}\prod_{i=1}^{d}\left(1-\frac{|r_i|}{L}\right)\gamma(r)(\rho^{|r|\bullet}-1)dr\right| \\
&\leq \int\limits_{[-L,L]^d} \|X_0\|_2^2|\rho^{dL}-1|dr \\
&\leq (2L)^d\|X_0\|_2^2\left(\frac{\varepsilon}{3(2L)^d\|X_0\|_2^2}\right) \\
&= \frac{\varepsilon}{3}
\end{aligned}
$$

for all $\rho \in [\rho_1, 1)$ and all $x \in \mathbb{R}^d$. Thus, with the triangle inequality and the result above, one has that for any $\rho \in [\rho_1, 1)$ and any $x \in \mathbb{R}^d$, $|f_\rho(x) - f(x)| \leq \varepsilon$. This implies the uniform convergence of f_ρ to f as $\rho \to 1^-$.

By (6.6), one can create a sequence $\{\rho_j\}_{j=1}^\infty$ (all in $(0,1)$) such that $\rho_j \to 1$ as $j \to \infty$ and $\|T - T^{(\rho(j))}\|_1 \leq 1/2^j$. Then for any j, $\|T^{(\rho(j))}\|_1 \leq \sum_{k=1}^\infty \|T^{(\rho(k))} - T\|_1 + \|T\|_1 \leq 1 + \|T\|_1$. Define

$$
G(x) := \sum_{j=1}^{\infty} |T^{(\rho(j))}(x) - T(x)| + T(x).
$$

This function is integrable since $\|T^{(\rho(j))} - T\|_1 \leq 1/2^j$ and $T(x)$ is integrable and non-negative. For any fixed j, notice that $T^{(\rho(j))}(x) \leq |T^{(\rho(j))}(x) - T(x)| + T(x) \leq G(x)$. Since $T(x) \leq G(x)$ trivially, $G(x)$ will be a dominating function for all of the $T^{(\rho(j))}$ and T. Let $A := A(\theta, d)$ be the constant from Lemma 5.2. Then by the definition of G, part (d) of Lemma 5.2 implies $f(x) \leq A{\cdot}G(x)$ and $f_{\rho(j)}(x) \leq A \cdot G(x)$. Lebesgue's dominated convergence theorem will then give $\int_{\mathbb{R}^d} |f_{\rho(j)}(x) - f(x)|dx \to 0$ as $j \to \infty$. In particular, for all $\nu \in \mathbb{R}^d$,

$$
\int_{\mathbb{R}^d} e^{ix\cdot\nu} f_{\rho(j)}(x)dx \quad\to\quad \int_{\mathbb{R}^d} e^{ix\cdot\nu} f(x)dx \tag{7.1}
$$

as $j \to \infty$. Again, refer to (A.1) and notice that for a given $x \in \mathbb{R}^d$,

$$\begin{aligned}
f_\rho(x) &= \lim_{L\to\infty} F(X^{(\rho,x)}, L) \\
&= \lim_{L\to\infty} L^{-d}\,\mathrm{E}\,|\,\mathrm{I}(X^{(\rho,x)}, L)|^2 \\
&= \lim_{L\to\infty} L^{-d} \int\limits_{[-L,L]^d} e^{-ix\cdot\nu}\left(\prod_{i=1}^{d}(L-|\nu_i|)\right)\gamma(\nu)\rho^{|\nu|_\bullet}d\nu \\
&= \lim_{L\to\infty} \int\limits_{[-L,L]^d} e^{-ix\cdot\nu}\prod_{i=1}^{d}\left(1-\frac{|\nu_i|}{L}\right)\gamma(\nu)\rho^{|\nu|_\bullet}d\nu \\
&= \lim_{L\to\infty} \int\limits_{\mathbb{R}^d} e^{-ix\cdot\nu}\mathbf{1}_{[-L,L]^d}(\nu)\cdot\prod_{j=1}^{d}\left(1-\frac{|\nu_j|}{L}\right)\gamma(\nu)\rho^{|\nu|_\bullet}d\nu. \qquad (7.2)
\end{aligned}$$

For each $\nu \in \mathbb{R}^d$, the integrand in (7.2) converges to $e^{-ix\cdot\nu}\gamma(\nu)\rho^{|\nu|_\bullet}$ as $L \to \infty$ and is dominated by $\gamma(0)\cdot\rho^{|\nu|_\bullet}$. Since $\rho^{|\nu|_\bullet}$ is integrable, Lebesgue's dominated convergence theorem gives

$$f_\rho(x) = \int_{\mathbb{R}^d} e^{-ix\cdot\nu}\gamma(\nu)\rho^{|\nu|_\bullet}d\nu.$$

Multiplying both sides of this equation by $(2\pi)^{-d/2}$, and using the fact that both f_ρ and $\gamma(\nu)\rho^{|\nu|_\bullet}$ are continuous and integrable, one can apply the inversion theorem and get that

$$\gamma(\nu)\rho^{|\nu|_\bullet} = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\nu}f_\rho(x)dx$$

for every $\nu \in \mathbb{R}^d$. Since $\gamma(\nu)\rho^{|\nu|_\bullet} \to \gamma(\nu)$ as $\rho \to 1^-$, (7.1) implies that

$$\gamma(\nu) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\nu}f(x)dx.$$

This shows that f is a spectral density for the random field X. Hence, the proof of Theorem 4.1 is complete.

APPENDIX A

Calculations

1. The set $H_{k,n}$

The following lemma was taken from Bradley [**6**].

Suppose $d \in \mathbb{N}$, and $k \in \mathbb{Z}^d$ are both fixed. For each $n \in \mathbb{N}$, let $H_{k,n} := \{(j, \ell) \in \{1, 2, \ldots, n\}^d \times \{1, 2, \ldots, n\}^d : j - \ell = k\}$.

LEMMA A.1. *For fixed $d \in \mathbb{N}$ and $k \in \mathbb{Z}^d$, $|H_{k,n}| \sim n^d$ as $n \to \infty$.*

Here, as in Chapter 1, $|S|$ denotes the cardinality of a set S.

PROOF. For each $j \in \{1, 2, \ldots, n\}^d$, there can be at most one element $\ell \in \{1, 2, \ldots, n\}^d$ such that $j - \ell = k$, hence $|H_{k,n}| \leq n^d$. Let $\|k\|_\infty := \max\{|k_1|, |k_2|, \ldots, |k_d|\}$. For $n > 2\|k\|_\infty$ and any $j \in \{1 + \|k\|_\infty, \ldots, n - \|k\|_\infty\}^d$, one has that $j - k \in \{1, 2, \ldots, n\}^d$ and therefore $(j, j - k) \in H_{k,n}$. Hence, for $n > 2\|k\|_\infty$, $\operatorname{card} H_{k,n} \geq (n - 2\|k\|_\infty)^d$. From these two inequalities, one gets the result. $\square$

2. The functions $T(x)$ and $F(X^{<x>}, L)$

Suppose $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field and $L \in (0, \infty)$. This section will first show the equality

$$\mathrm{E}\left|\int_{[0,L]^d} e^{-ix\cdot\nu} X_\nu \, d\nu\right|^2 = \int_{[-L,L]^d} e^{-ix\cdot\nu} \left(\prod_{j=1}^{d} (L - |\nu_j|)\right) \gamma(\nu) d\nu, \tag{A.1}$$

where $L > 0$, and $x \in \mathbb{R}^d$. Notice that for $L = 1$, this is the function $T(x)$ from chapter 4, and by multiplying by L^{-d}, it is the function $F(X^{<x>}, L)$ from chapter 5. First, it will be done for $d = 1$, and then be easily extended to general d.

For the case $d = 1$, use Fubini to note the following equality:

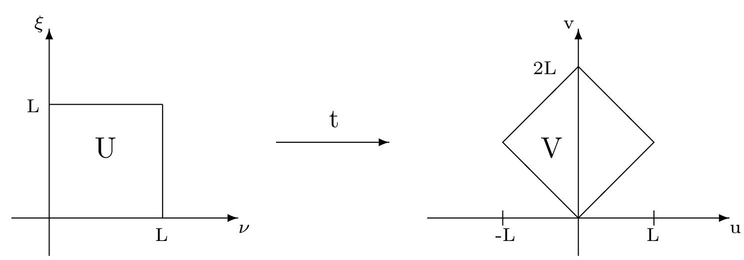
$$\begin{aligned}
\mathrm{E}\left|\int_0^L e^{-ix\nu} X_\nu \, d\nu\right|^2 &= \mathrm{E}\left(\int_0^L e^{-ix\nu} X_\nu d\nu\right)\left(\int_0^L \overline{e^{-ix\xi} X_\xi} d\xi\right) \\
&= \mathrm{E}\left(\int_0^L \int_0^L e^{-ix(\nu-\xi)} X_\nu \overline{X_\xi} d\xi d\nu\right) \\
&= \int_0^L \int_0^L e^{-ix(\nu-\xi)}\, \mathrm{E}(X_\nu \overline{X_\xi}) d\xi d\nu \\
&= \int_0^L \int_0^L e^{-ix(\nu-\xi)} \gamma(\nu-\xi) d\xi d\nu. \qquad \text{(A.2)}
\end{aligned}$$

Let $f(\nu,\xi) := e^{-ix(\nu-\xi)} \cdot \gamma(\nu-\xi)$, and let $g(u,v) := e^{-ixu} \cdot \gamma(u)$. Then define the transformation $t : \mathbb{R}^2 \mapsto \mathbb{R}^2$ by $t(\nu,\xi) = (u,v)$ where $u = \nu - \xi$ and $v = \xi + \nu$. Notice that $f(\nu,\xi) = g(t(\nu,\xi))$. The Jacobian is given by

$$J(\nu,\xi) = \det \begin{bmatrix} \partial u/\partial\nu & \partial u/\partial\xi \\ \partial v/\partial\nu & \partial v/\partial\xi \end{bmatrix} = 2.$$

Let $U = [0,L]^2$, so that $V := t(U)$ is the square in $\mathbb{R}^2$ with corners $(0,0)$, (L,L), $(0,2L)$, and $(-L,L)$.

FIGURE 1. The transformation t



Using Theorem 10.9 from Rudin [**17**] or Theorem 17.2 from Billingsley [**1**] (these theorems can extend to complex functions by taking the real and imaginary parts separately),

$$\begin{aligned}
\int_0^L \int_0^L e^{-ix(\nu-\xi)} \gamma(\nu-\xi) d\xi d\nu &= \int_U f(\nu,\xi) d\xi d\nu \\
&= \frac{1}{2} \int_U g(t(\nu,\xi)) |J(\nu,\xi)| d\xi d\nu \\
&= \frac{1}{2} \int_V g(u,v) dv du.
\end{aligned}$$

Since $g(u,v)$ depends only on u, it can be written as $g(u)$. Use Figure 1 above to see that

$$\begin{aligned}
\frac{1}{2}\int_V g(u,v)dvdu &= \frac{1}{2}\int_{-L}^{0}\int_{-u}^{2L+u} g(u)dvdu + \frac{1}{2}\int_{0}^{L}\int_{u}^{2L-u} g(u)dvdu \\
&= \frac{1}{2}\int_{-L}^{0} g(u)(2L+2u)du + \frac{1}{2}\int_{0}^{L} g(u)(2L-2u)du \\
&= \int_{-L}^{L} g(u)(L-|u|)du \\
&= \int_{-L}^{L} e^{-ixu}\gamma(u)(L-|u|)du \qquad \text{(A.3)}
\end{aligned}$$

Thus, since (A.2) and (A.3) are equal,

$$\mathrm{E}\left|\int_0^L e^{-ix\nu} X_\nu\, d\nu\right|^2 = \int_{-L}^{L} e^{-ix\nu}(L-|\nu|)\gamma(\nu)d\nu \qquad \text{(A.4)}$$

With an adaptation of the argument that the terms in (A.2) and (A.3) are equal, one can readily see that for any continuous function $h : \mathbb{R} \to \mathbb{C}$, and any $L > 0$,

$$\int_0^L \int_0^L h(\nu-\xi)d\xi d\nu = \int_{-L}^{L} h(\nu)(L-|\nu|)d\nu. \qquad \text{(A.5)}$$

Now for general $d > 0$, note that (similar to the calculation leading to (A.2))

$$\begin{aligned}
&\mathrm{E}\left|\int\limits_{[0,L]^d} e^{-ix\cdot\nu} X_\nu d\nu\right|^2 \\
&= \mathrm{E}\left(\int\limits_{[0,L]^d} e^{-ix\cdot\nu} X_\nu d\nu\right)\left(\int\limits_{[0,L]^d} e^{ix\cdot\xi}\overline{X}_\xi d\xi\right) \\
&= \mathrm{E}\left(\int\limits_{[0,L]^d}\int\limits_{[0,L]^d} e^{-ix\cdot(\nu-\xi)} X_\nu \overline{X}_\xi d\xi d\nu\right) \\
&= \int\limits_{[0,L]^d}\int\limits_{[0,L]^d} e^{-ix\cdot(\nu-\xi)}\gamma(\nu-\xi)d\xi d\nu \\
&= \int\limits_{[0,L]^2} e^{-ix_1(\nu_1-\xi_1)}\int\limits_{[0,L]^2} e^{-ix_2(\nu_2-\xi_2)}\cdots\int\limits_{[0,L]^2} e^{-ix_d(\nu_d-\xi_d)}\gamma(\nu-\xi)d\xi_d d\nu_d\cdots d\xi_1 d\nu_1.
\end{aligned}$$

Using (A.5) on the inner most integral, one obtains

$$\int\limits_{[0,L]^2} e^{-ix_d(\nu_d-\xi_d)}\gamma(\nu-\xi)d\xi_d d\nu_d = \int\limits_{[-L,L]} e^{-ix_d\mu_d}(L-|\mu_d|)\gamma(\nu_1-\xi_1,,\ldots,\nu_{d-1}-\xi_{d-1},\mu_d)d\mu_d.$$

Repeating this $d-1$ more times, (A.1) is obtained.

Lemma A.2. *For any fixed $L > 0$, the mapping*

$$x \mapsto \mathrm{E}\left| \int\limits_{[0,L]^d} e^{-ix\cdot\nu} X_\nu d\nu \right|^2, \; x \in \mathbb{R}^d \tag{A.6}$$

is uniformly continuous on $\mathbb{R}^d$.

Proof. The proof is trivial in the degenerate case $\gamma(0) = 0$ ($X_\nu = 0$ almost surely for all $\nu \in \mathbb{R}^d$), so assume that $\gamma(0) > 0$. First, recall that for $\phi, \psi \in \mathbb{R}$, one has $|e^{i\phi} - e^{i\psi}| \leq |\phi - \psi|$. Suppose $x, y \in \mathbb{R}^d$. Then by (A.1) and the calculation above,

$$\begin{aligned}
&\left| \mathrm{E}\left| \int\limits_{[0,L]^d} e^{-ix\cdot\nu} X_\nu d\nu \right|^2 - \mathrm{E}\left| \int\limits_{[0,L]^d} e^{-iy\cdot\nu} X_\nu d\nu \right|^2 \right| \\
&= \left| \int\limits_{[-L,L]^d} e^{-ix\cdot\nu} \left(\prod_{j=1}^d (L - |\nu_j|) \right) \gamma(\nu) d\nu - \int\limits_{[-L,L]^d} e^{-iy\cdot\nu} \left(\prod_{j=1}^d (L - |\nu_j|) \right) \gamma(\nu) d\nu \right| \\
&\leq \int\limits_{[-L,L]^d} |e^{-ix\cdot\nu} - e^{-iy\cdot\nu}| \left(\prod_{j=1}^d (L - |\nu_j|) \right) |\gamma(\nu)| d\nu \\
&\leq \int\limits_{[-L,L]^d} |x\cdot\nu - y\cdot\nu| \cdot L^d \gamma(0) d\nu \\
&\leq \int\limits_{[-L,L]^d} \|x - y\| \cdot \|\nu\| \cdot L^d \gamma(0) d\nu \\
&\leq \|x - y\| \cdot (2L)^d (\sqrt{d} L) L^d \gamma(0)
\end{aligned} \tag{A.7}$$

For any given $\varepsilon > 0$, if $\|x - y\| < \varepsilon/(2^d \sqrt{d} L^{2d+1} \gamma(0))$ then the first term in (A.7) is less than ε. Hence, (A.6) is a uniformly continuous mapping. $\square$

3. Converging Covariances

Lemma A.3. *Suppose that Y and Z are complex valued random variables such that $\|Y\|_2 < \infty$, $\|Z\|_2 < \infty$, and that for every $L \in \mathbb{N}$, Y_L and Z_L are complex valued random variables with $\|Y_L\|_2 < \infty$ and $\|Z_L\|_2 < \infty$. Suppose that $\|Y_L - Y\|_2 \to 0$ and $\|Z_L - Z\|_2 \to 0$ as $L \to \infty$. Then*

$$\mathrm{E}\, Y_L \overline{Z}_L \to \mathrm{E}\, Y \overline{Z}$$

as $L \to \infty$.

PROOF. Note that for $L \geq 1$, $|\|Y\|_2 - \|Y_L\|_2| \leq \|Y - Y_L\|_2$. Therefore $\|Y_L\|_2 \to \|Y\|_2$ as $L \to \infty$. Now, since

$$\begin{aligned}
\left|\mathrm{E}\, Y_L \overline{Z_L} - \mathrm{E}\, Y \overline{Z}\right| &= \left|\mathrm{E}\, Y_L \overline{Z_L} - \mathrm{E}\, Y_L \overline{Z} + \mathrm{E}\, Y_L \overline{Z} - \mathrm{E}\, Y \overline{Z}\right| \\
&\leq \left|\mathrm{E}\, Y_L \overline{Z_L} - \mathrm{E}\, Y_L \overline{Z}\right| + \left|\mathrm{E}\, Y_L \overline{Z} - \mathrm{E}\, Y \overline{Z}\right| \\
&= \left|\mathrm{E}(Y_L(\overline{Z_L - Z}))\right| + \left|\mathrm{E}((Y_L - Y)\overline{Z})\right| \\
&\leq \|Y_L\|_2 \cdot \|Z_L - Z\|_2 + \|Y_L - Y\|_2 \cdot \|Z\|_2,
\end{aligned}$$

one can see that $|\,\mathrm{E}\, Y_L \overline{Z}_L - \mathrm{E}\, Y \overline{Z}| \to 0$ as $L \to \infty$. Hence, the proof is complete. □

4. Integral Version of Minkowski's Inequality

LEMMA A.4. *Suppose that Q is a Borel subset of $\mathbb{R}^d$ and that $(X_t : t \in \mathbb{R}^d)$ is a complex valued random field on a probability space $(\Omega, \mathcal{F}, P)$. If $\int_Q \|X_t\|_2 dt < \infty$, then $\int_Q |X_t| dt < \infty$ almost surely and*

$$\left\| \int_Q X_t dt \right\|_2 \leq \int_Q \|X_t\|_2 dt.$$

PROOF. Since $\|X_t\|_1 \leq \|X_t\|_2$ for all t, then $\mathrm{E}\left(\int_Q |X_t| dt\right) = \int_Q \mathrm{E}\,|X_t| dt \leq \int_Q \|X_t\|_2 dt < \infty$, and hence $\int_Q |X_t| dt < \infty$ almost surely. Thus,

$$\begin{aligned}
\mathrm{E}\left|\int_Q X_t dt\right|^2 &\leq \mathrm{E}\left(\int_Q |X_t| dt\right)^2 \\
&= \mathrm{E}\left(\int_Q |X_t| dt\right)\left(\int_Q |X_u| du\right) \\
&= \mathrm{E}\left(\int_Q \int_Q |X_t| \cdot |X_u| du dt\right) \\
&= \int_Q \int_Q \mathrm{E}\left(|X_t| \cdot |X_u|\right) du dt \\
&\leq \int_Q \int_Q \|X_t\|_2 \|X_u\|_2 du dt \\
&= \left(\int_Q \|X_t\|_2 dt\right) \cdot \left(\int_Q \|X_u\|_2 du\right) \\
&= \left(\int_Q \|X_t\|_2 dt\right)^2.
\end{aligned}$$

Taking square roots of the both sides now gives the desired inequality.

□

5. Enlarging the probability space

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space, and $X := (X_r : r \in \mathbb{R}^d)$ is a CCWS random field (see Definition 4.1 and the paragraph before Definition 1.1). Consider the product probability space $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ defined by the Cartesian product $\boldsymbol{\Omega} := \Omega \times \Omega \times \ldots$, the product σ-field $\boldsymbol{\mathcal{F}} := \mathcal{F} \times \mathcal{F} \times \ldots$, and the product probability measure $\mathbf{P} := P \times P \times \ldots$ (refer to Theorem 8.2.2 in [**10**]). Define on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ the random field $\widetilde{X} := (\widetilde{X}_r : r \in \mathbb{R}^d)$ and the random fields $W^n := (W^n_r : r \in \mathbb{R}^d)$ for each $n \in \mathbb{N}$ by $\widetilde{X}_r(\tilde{\omega}) := X_r(\omega_0)$, and $W^n_r(\tilde{\omega}) := X_r(\omega_n)$ for every $r \in \mathbb{R}^d$ and $\tilde{\omega} := (\omega_0, \omega_1, \omega_2, \ldots) \in \boldsymbol{\Omega}$.

For any finite sequence $r_1, r_2, \ldots, r_J$ in $\mathbb{R}^d$, any finite sequence $Q_1, Q_2, \ldots, Q_L$ of bounded Borel subsets of $\mathbb{R}^d$, and any finite sequence $g_1, g_2, \ldots, g_L$ of bounded, Borel, complex-valued functions on $\mathbb{R}^d$, the random vector $V := ((X_{r(j)} : j = 1, 2, \ldots, J), (\int_{Q(\ell)} g_\ell(\nu) X_\nu d\nu : \ell = 1, 2, \ldots, L))$ defined on $(\Omega, \mathcal{F}, P)$, the random vector $\widetilde{V} := ((\widetilde{X}_{r(j)} : j = 1, 2, \ldots, J), (\int_{Q(\ell)} g_\ell(\nu) \widetilde{X}_\nu d\nu : \ell = 1, 2, \ldots, L))$ defined on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$, and the random vectors $U^n := ((W^n_{r(j)} : j = 1, 2, \ldots, J), (\int_{Q(\ell)} g_\ell(\nu) W^n_\nu d\nu : \ell = 1, 2, \ldots, L))$ also defined on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$, are all identically distributed on $\mathbb{C}^{J+L}$. It is also easy to see that the random vectors $\widetilde{V}, U^1, U^2, U^3, \ldots$ are independent.

By an elementary argument, the random field X on $(\Omega, \mathcal{F}, P)$ and the random fields $\widetilde{X}, W^1, W^2, W^3, \ldots$ on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ have the same distribution as described in the previous paragraph.

This process can also be done for any countable index set instead of $\mathbb{N}$ (for the index n), and in particular, $\mathbb{Z}^d$.

Constructing $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ is called "enlarging" the probability space $(\Omega, \mathcal{F}, P)$. When one does this, it is customary (with a slight abuse of notation) to refer to the enlarged probability space $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ by the original one $(\Omega, \mathcal{F}, P)$, and to also refer to the random field $\widetilde{X}$ by the original X.

APPENDIX B

Fourier Analysis

1. Basic Definitions and Theorems

Most of the techniques used are taken from Chapter 9 in Rudin [**16**] and Chapter 7 in Rudin [**18**]. To simplify the appearance of some calculations ahead, let $\mu_d(\cdot)$ denote a rescaled Lebesgue measure on $\mathbb{R}^d$ defined by $d\mu_d(x) = (2\pi)^{-d/2}dx$. Note the difference between $dm_d(x) = (2\pi)^{-d}dx$, defined in Chapter 1, and $d\mu_d(x)$. The convolution of two functions on $\mathbb{R}^d$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)d\mu_d(y),$$

as long as the integral exists. The Fourier transform $\hat{f} : \mathbb{R}^d \to \mathbb{R}$ of a function $f \in L^1(\mathbb{R}^d)$ will be given by

$$\hat{f}(\nu) = \int_{\mathbb{R}^d} e^{i\nu\cdot x} f(x)d\mu_d(x).$$

This is not standard. In most texts, $e^{i\nu\cdot x}$ would be replaced by $e^{-i\nu\cdot x}$ in the definition above. The theory is the same and makes the arguments in Chapter 6 easier to follow.

THEOREM B.1. *Suppose $f, g \in L^1(\mathbb{R}^d)$. Then $f * g \in L^1(\mathbb{R}^d)$ and $\widehat{f * g} = \hat{f}\hat{g}$.*

THEOREM B.2. *If $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1(\mathbb{R}^d)$, then*

$$f(x) = \int_{\mathbb{R}^d} e^{-ix\cdot\nu} \hat{f}(\nu)d\mu_d(\nu)$$

for almost every $x \in \mathbb{R}^d$. If f is also assumed to be continuous, then the equality holds for all $x \in \mathbb{R}^d$.

This is taken from part (c) of Section 7.7 in Rudin [**18**].

2. The Even Functions H and h_λ

For $t, x \in \mathbb{R}^d$, let $|t|_\bullet = \sum_{i=1}^d |t_i|$ and $t \cdot x = \sum_{i=1}^d t_i x_i$. Define $H : \mathbb{R}^d \to \mathbb{R}$ by

$$H(t) = e^{-|t|_\bullet},$$

and then let

$$h_\lambda(x) = \int_{\mathbb{R}^d} H(\lambda t)e^{it\cdot x}d\mu_d(t) \qquad (\lambda > 0).$$

Since H is an even function, notice that $h_\lambda(x) = \int_{\mathbb{R}^d} H(\lambda t) e^{-it\cdot x} d\mu_d(t)$ as well. A simple computation will give

$$h_\lambda(x) = \left(\frac{2}{\pi}\right)^{d/2} \prod_{i=1}^{d} \frac{\lambda}{\lambda^2 + x_i^2},$$

and hence, $\int_{\mathbb{R}^d} h_\lambda(x) d\mu_d(x) = 1$. By Theorem B.2, $H(\lambda t) = \int_{\mathbb{R}^d} e^{-it\cdot\nu} h_\lambda(\nu) d\mu_d(\nu)$ for every $t \in \mathbb{R}^d$. Again, since $h_\lambda(\nu)$ is an even function, one also has

$$H(\lambda t) = \int_{\mathbb{R}^d} e^{it\cdot\nu} h_\lambda(\nu) d\mu_d(\nu)$$

for every $t \in \mathbb{R}^d$, and therefore, $H(\lambda t) = \hat{h}_\lambda(t)$.

THEOREM B.3. *If $f \in L^1(\mathbb{R}^d)$, then*

$$\lim_{\lambda \to 0^+} \|f * h_\lambda - f\|_1 = 0.$$

For $d = 1$, this was done in Theorem 9.10 of Rudin [**16**]. The argument for general d is analogous.

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